ПРОЦЕССЫ УПРАВЛЕНИЯ

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The differential game with inertial players under integral constraints on controls

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We consider pursuit-evasion differential games between inertial players (a pursuer and an evader) whose controls are subject to integral constraints. The pursuer is believed to have captured the evader when their positions coincide. The key method used to provide a win for the pursuer in the pursuit differential game is the parallel pursuit strategy (in brief, the Π-strategy). We obtain sufficient conditions for the solvability of pursuit-evasion problems. Furthermore, we investigate Isaacs' "life-line" game in favor of the pursuer in the case of the identical initial velocities of the players. Here, the main lemma characterizing its monotonicity property provides an analytical formula for the players' meeting domain. This paper extends and continues the works of R. Isaacs, L. A. Petrosyan, B. N. Pshenichnyi, N. Yu. Satimov, the authors of this article, and other researchers.

Keywords: differential game, pursuer, evader, strategy, "life-line" game.

1. Introduction. Differential games constitute a special class of problems for conflict-controlled dynamic systems described by differential equations. In 1965, the concept of "differential game" was first introduced by American mathematician R. Isaacs [1]. From then onward, L. S. Pontryagin [2], N. N. Krasovskii [3], L. A. Petrosyan [4–6], A. Friedman [7], N. N. Krasovskii and A. I. Subbotin [8], L. D. Berkovitz [9], B. N. Pshenichnyi [10], A. I. Subbotin [11], A. A. Chikrii [12], J. Lewin [13], N. Yu. Satimov [14], and many other researchers developed the ideas of Isaacs and contributed to the development of differential game theory.

Differential games have a distinctive interest in pursuit-evasion problems due to several specific features, such as the complexity of problem statements and their applications to practice (see e.g. [1–14]). Notably, Isaacs applied the method of characteristics to the Hamilton — Jacobi equation to solve differential games. This method demands strict conditions on the data of a game and can only be justified in special cases. Therefore, examining concrete examples in the theory of differential games is particularly important. One of such examples is known as Isaacs' "life-line" game with simple player dynamics. R. Isaacs studied this game when the life-line subset is a half plane [1, Problem 9.5.1]. At a later time, L. A. Petrosyan [4, 5] considered the "life-line" game in a more general context using an exclusive strategy referred to as a parallel pursuit strategy (or, briefly the Π-strategy). Using the support function of the set-valued map, A. Azamov solved the "life-line" game analytically for the case of multiple pursuers and one evader when the players have different speeds [15]. Afterwards, applying the Π-strategy in pursuit problems and development of the "life-line" game can be traced in [12, 16–19].

Pursuit-evasion problems in the differential game with integral constraints on players' controls have been considered in several works [14, 16, 18, 20]. Studying the problems associated with inertial players is more intriguing and more complex than simple pursuit differential games. Furthermore, there are many applications of differential games, for example, in motions of ships, bathyscaphes, missiles, drones, and others [2, 8, 14, 21–23]. B. T. Samatov et al. [23] studied pursuit-evasion differential games with inertial players under geometric constraints on controls, and they also looked at Isaacs' "life-line" problem from the pursuer's point of view.

In this paper, we analyze pursuit-evasion problems in a differential game with inertial players (a pursuer and an evader) whose controls are subject to integral constraints. The main tool used to provide winning for the pursuer in the pursuit problems and in the "life-line" problem (for the case of equal initial velocities of the players) is the Π-strategy. Using this strategy, necessary and sufficient conditions for completing the pursuit problems are obtained, and a set of meeting points for the players is constructed. When solving the "life-line" problem for the pursuer, the main lemma regarding the monotone decrease (by inclusion) of the players' reachability set over time is proved. Furthermore, illustrative example are given for the meeting domain of the players.

2. Statement of the problems. We study a differential game with two players, called *the pursuer* and *the evader*, in space $\mathbb{R}^n$.

Let a parameter $x \in \mathbb{R}^n$ (respectively, a parameter $y \in \mathbb{R}^n$) designate a position of the pursuer (the evader). Then, assume the pursuer begins its motion from an initial position x_0 and with an initial velocity x_1 in accordance with the dynamics

$$\ddot{x} = u, \quad x(0) = x_0, \quad \dot{x}(0) = x_1, \tag{1}$$

where the parameter $u \in \mathbb{R}^n$ is the acceleration vector that serves as the control of the pursuer. Ultimately, the temporal change of the vector u must be a measurable function $u(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$.

Definition 1. A measurable function $u(\cdot) = (u(t), t \geqslant 0)$ is called an admissible control for the pursuer, if the integral constraint

$$\int_0^t (t-s)|u(s)|^2 ds \leqslant \rho_0^2, \quad \rho_0 > 0, \qquad t \geqslant 0, \tag{2}$$

holds.

From now on, the class consisting of all admissible controls $u(\cdot)$ for the pursuer will be denoted by $\mathbf{U}_I$.

Similar to the above, the evader begins its motion from an initial position y_0 and with an initial velocity y_1 in accord to the dynamics

$$\ddot{y} = v, \quad y(0) = y_0, \quad \dot{y}(0) = y_1, \tag{3}$$

here the parameter $v \in \mathbb{R}^n$ is the acceleration vector that serves as the control of the evader. Similarly, the temporal change of the vector v must be a measurable function $v(\cdot) : \mathbb{R}_+ \to \mathbb{R}^n$.

Definition 2. A measurable function $v(\cdot) = (v(t), t \geqslant 0)$ is called an admissible control for the evader, if the integral constraint

$$\int\limits_0^t (t-s)|v(s)|^2 ds \leqslant \sigma_0^2, \quad \sigma_0 > 0, \quad t \geqslant 0, \tag{4}$$

holds.

Here and subsequently, the class consisting of all admissible controls $v(\cdot)$ for the evader will be denoted by $\mathbf{V}_I$.

The number ρ_0^2 (respectively, σ_0^2) embodies the maximum quantity of a resource of the pursuer (respectively, the evader).

For the sake of simplifying the calculations, we will introduce the following reductions: $z = x - y, z_0 = x_0 - y_0, z_1 = x_1 - y_1$. Then systems (1) and (3) reduce to the form

$$\ddot{z} = u - v, \quad z(0) = z_0, \quad \dot{z}(0) = z_1. \tag{5}$$

Summing up from the new denotation introduced above, we can express the main goals of the players, that is, the primary objective of the pursuer is to catch the evader, i.e., to specifically achieve the equality $z(t) = 0$ from the given initial values z_0 and z_1 in the shortest time t. For the evader, the main goal is to maintain the relation $z(t) \neq 0$ for each $t \in [0, +\infty)$, and if this is impossible, then put back the occurrence of capture. Additionally, we suppose that $z_0 \neq 0$ at the start of the game.

Definition 3. A map $\boldsymbol{u} : \mathbf{V}_I \to \mathbf{U}_I$ is said to be a strategy of the pursuer, if the following conditions are fulfilled:

a) (Admissibility) for each control $v(\cdot) \in \mathbf{V}_I$, the inclusion $\boldsymbol{u}(v(\cdot)) \in \mathbf{U}_I$ is satisfied;

b) (Volterranianity) for every $v_1(\cdot), v_2(\cdot) \in \mathbf{V}_I$ and $t, t \geqslant 0$, the equality $v_1(s) = v_2(s)$ a.e. on $[0,t]$ implies $u_1(s) = u_2(s)$ a.e. on $[0,t]$ with $u_i(\cdot) = \boldsymbol{u}(v_i(\cdot)),\ i = 1,\ 2$.

Definition 4. We call a strategy $\boldsymbol{u}(v(\cdot))$ the parallel convergence strategy (Π-strategy), if for arbitrary $v(\cdot) \in \mathbf{V}_I$ the solution $z(t)$ of the initial value problem

$$\ddot{z} = \boldsymbol{u}(v(t)) - v(t), \quad z(0) = z_0, \quad \dot{z}(0) = z_1,$$

can be expressed as

$$z(t) = z_0 \Gamma(t, v(\cdot)), \quad \Gamma(0, v(\cdot)) = 1, \quad t \geqslant 0,$$

where $\Gamma(t, v(\cdot))$ is a scalar function.

Obviously, $\Gamma(t, v(\cdot))$ also satisfies the Volterra property in Definition 3.

Definition 5. It is said that the pursuer wins by using the Π-strategy on a finite time interval $[0, T]$, if for any $v(\cdot) \in \mathbf{V}_I$,

a) $z(\theta) = 0$ at some instant $\theta \in [0, T]$;
b) $\widetilde{u}(\cdot) \in \mathbf{U}_I$, where

$$\widetilde{u}(t) = \begin{cases} \boldsymbol{u}(v(t)), & \text{if } \ 0 \leqslant t \leqslant \theta, \\ 0, & \text{if } \ t > \theta. \end{cases}$$

The number T is called a guaranteed capture time in the pursuit problem.

This paper is dedicated to investigating the following games where the controls $u(\cdot)$ and $v(\cdot)$ of the players are subject to the integral constraints:

Game 1. Solve pursuit-evasion problems in the differential game (1)–(4) for the case of linear dependence of the vectors, which are the difference in the initial states and the difference in the initial velocities of the players.

Game 2. Solve the differential game with "life-line" in the case of the identical initial velocities of the players.

3. Pursuit-evasion problems in Game 1. In this section, we consider the pursuit-evasion problems in the differential game (1)–(4) for the case where the vectors z_0 and z_1 are collinear, which means there is a finite number k such that

$$z_1 = kz_0, \quad k \in \mathbb{R} \setminus \{0\}. \tag{6}$$

Here we assume that k is not equal to zero, because in the next section we will consider the pursuit problem for the case of $k = 0$. As a result of (6), when choosing admissible controls $u(\cdot) \in \mathbf{U}_I$ and $v(\cdot) \in \mathbf{V}_I$, the solution $z(t)$ of the initial value problem (5) will have form

$$z(t) = z_0 + kz_0 t + \int\limits_0^t (t - s)(u(s) - v(s))ds. \tag{7}$$

3.1. Implementing a* Π*-strategy to the pursuit problem. Next, we will define the Π-strategy based on the existing works [5, 6, 10, 15, 16] and take a sufficient condition for the occurrence of pursuit in the case (6). To construct the Π-strategy, suppose that the pursuer is aware of the initial date z_0, ρ_0, σ_0, the finite number k, and the value $v(t)$ at the current time t.

Let's introduce the following new constants for simplicity of our calculation:

$$\delta = \frac{\rho_0^2 - \sigma_0^2}{|z_0|}, \quad \mu = \frac{\rho_0 - \sigma_0}{\sqrt{2}|z_0|}, \quad a = -2k\delta|z_0|(\mu - k), \quad b = 2\delta|z_0|(\mu - k)(2\mu - k),$$

and let's set

$$\gamma_k(v) = \begin{cases} \gamma_*(v), & \text{if } \ \rho_0 \geqslant \sigma_0 \text{ and } k < 0, \\ \gamma^*(v), & \text{if } \ \rho_0 > \sigma_0 \text{ and } 0 < k < \mu, \end{cases} \tag{8}$$

where

$$\gamma_*(v) = \frac{\delta}{2} + \langle v, \xi_0 \rangle + \sqrt{\left(\frac{\delta}{2} + \langle v, \xi_0 \rangle\right)^2 + a}, \tag{9}$$

$$\gamma^*(v) = -\frac{\delta}{2} + \langle v, \xi_0 \rangle + \sqrt{\left(-\frac{\delta}{2} + \langle v, \xi_0 \rangle\right)^2 + b}, \tag{10}$$

and $\xi_0 = z_0/|z_0|$, $\langle v, \xi_0 \rangle$ is the inner product of the vectors v and ξ_0 in $\mathbb{R}^n$.

Definition 6. The control function

$$\boldsymbol{u}_k(v) = v - \gamma_k(v)\xi_0, \tag{11}$$

is called the Π-strategy of the pursuer in the pursuit problem of Game 1.

In general, the scalar function $\gamma_k(v)$ is termed *the resolving function* in the pursuit problem. Now, we present the following important property for strategy (11) and resolving function (8).

Lemma 1. *Let $\rho_0 \geqslant \sigma_0$ and $k \in (-\infty, \mu)$ be satisfied. Then the function $\gamma_k(v)$ is well-defined, non-negative for any control $v(\cdot) \in \mathbf{V}_I$ and, what's more, the equality*

$$|\boldsymbol{u}_k(v)|^2 = \begin{cases} |v|^2 + \delta\gamma_k(v) + a, & \textit{if} \;\; \rho_0 \geqslant \sigma_0 \; \textit{and} \; k < 0, \\ |v|^2 - \delta\gamma_k(v) + b, & \textit{if} \;\; \rho_0 > \sigma_0 \; \textit{and} \; 0 < k < \mu, \end{cases} \tag{12}$$

is met.

Let

$$\Gamma_k(t) = \begin{cases} \Gamma_*(t), & \text{if} \;\; \rho_0 \geqslant \sigma_0 \text{ and } k < 0, \\ \Gamma^*(t), & \text{if} \;\; \rho_0 > \sigma_0 \text{ and } 0 < k < \mu, \end{cases}$$

where

$$\Gamma_*(t) = 1 + kt - \frac{t}{|z_0|}\left[\frac{\delta}{4}t - \frac{\sigma_0}{\sqrt{2}} + \sqrt{\left(\frac{\delta}{4}t - \frac{\sigma_0}{\sqrt{2}}\right)^2 + \frac{a}{4}t^2}\right]$$

and

$$\Gamma^*(t) = 1 + kt + \frac{t}{|z_0|}\left[\frac{\delta}{4}t + \frac{\sigma_0}{\sqrt{2}} - \sqrt{\left(\frac{\delta}{4}t + \frac{\sigma_0}{\sqrt{2}}\right)^2 + \frac{b}{4}t^2}\right].$$

Proposition 1. If $\rho_0 \geqslant \sigma_0$ and $k \in (-\infty, \mu)$, then there exists at least positive root of the equation

$$\Gamma_k(t) = 0,$$

and we denote by T_k the smallest positive root, where $T_k = 1/(\mu - k)$.

Consider the scalar function

$$\bar{\Gamma}_k(t, v(\cdot)) = 1 + kt - \frac{1}{|z_0|}\int_0^t (t-s)\gamma_k(v(s))ds \tag{13}$$

with respect to t, $t \geqslant 0$. Usually, the scalar function $\bar{\Gamma}_k(t, v(\cdot))$ is called the *convergence function* of the players in the pursuit problem for any $v(\cdot) \in \mathbf{V}_I$.

Lemma 2. *Let Proposition 1 be satisfied. Then the convergence function* (13) *is bounded on time interval* $[0, T_k]$ *as follows:*

$$0 < \bar{\Gamma}_k(t, v(\cdot)) \leqslant \Gamma_k(t).$$

P r o o f. Clearly, $\bar{\Gamma}_k(0, v(\cdot)) = 1$.

a). First of all, suppose that $\rho_0 \geqslant \sigma_0$ and $k < 0$, then the convergence function (13) is monotonically decreasing in t, $t \geqslant 0$. From the form (9), it is not difficult to show that $\gamma_*(v)$ is increasing with $\langle v, \xi_0\rangle$. For this convergence function, we may write estimates

$$\bar{\Gamma}_k(t, v(\cdot)) \leqslant 1 + kt - \frac{1}{|z_0|}\min_{v(\cdot)\in\mathbf{V}_I}\int_0^t (t-s)\gamma_*(v(s))ds \leqslant$$

$$\leqslant 1 + kt - \frac{1}{|z_0|} \int_0^t (t-s) \left(\frac{\delta}{2} - |v(s)| + \sqrt{\left(\frac{\delta}{2} - |v(s)| \right)^2 + a} \right) ds,$$

or, in the short form

$$\bar{\Gamma}_k(t, v(\cdot)) \leqslant 1 + kt - \frac{1}{|z_0|} \min_{v(\cdot) \in \mathbf{V}_I} \int_0^t \varphi(s) \psi\big(\varrho(s)\big) ds, \tag{14}$$

where $\psi\big(\varrho(s)\big) = \varrho(s) + \sqrt{\varrho^2(s) + a}$, $\varrho(s) = \delta/2 - |v(s)|$ and $\varphi(s) = t - s$ ($s \in [0, t]$). Here, $\varphi(s)$ is a non-negative and continuous function. Thus, by the convexity of $\psi\big(\varrho(s)\big)$, we can use the Jensen inequality for the integral in (14), i.e.,

$$\int_0^t \varphi(s) \psi(\varrho(s)) ds \geqslant \int_0^t \varphi(s) ds \psi \left(\frac{\int_0^t \varphi(s) \varrho(s) ds}{\int_0^t \varphi(s) ds} \right). \tag{15}$$

As a consequence of (15), the right side of (14) takes the form

$$\bar{\Gamma}_k(t, v(\cdot)) \leqslant 1 + kt - \frac{1}{|z_0|} f(t, \omega), \tag{16}$$

here $f(t, \omega) = \omega + \sqrt{\omega^2 + t^4 a/4}$ and $\omega = \delta t^2/4 - \int_0^t (t-s)|v(s)| ds$. It is clear that $f(t, \omega)$ is a monotonically increasing function with respect to ω. Considering the Cauchy — Schwartz inequality and taking account of the constraint (4)

$$\int_0^t (t-s)|v(s)| ds \leqslant \left(\int_0^t (t-s) ds \right)^{1/2} \left(\int_0^t (t-s)|v(s)|^2 ds \right)^{1/2} \leqslant \frac{\sigma_0}{\sqrt{2}} t \tag{17}$$

for ω, we can estimate the function $f(t, \omega)$ in the form

$$f(t, \omega) \geqslant \omega_{\min} + \sqrt{\omega_{\min}^2 + \frac{a}{4} t^4} = \frac{\delta}{4} t^2 - \frac{\sigma_0}{\sqrt{2}} t + \sqrt{\left(\frac{\delta}{4} t^2 - \frac{\sigma_0}{\sqrt{2}} t \right)^2 + \frac{a}{4} t^4}.$$

From an estimation of the function $f(t, \omega)$, we obtain the results for the right side of (16):

$$\bar{\Gamma}_k(t, v(\cdot)) \leqslant 1 + kt - \frac{t}{|z_0|} \left[\frac{\delta}{4} t - \frac{\sigma_0}{\sqrt{2}} + \sqrt{\left(\frac{\delta}{4} t - \frac{\sigma_0}{\sqrt{2}} \right)^2 + \frac{a}{4} t^2} \right],$$

or

$$\bar{\Gamma}_k(t, v(\cdot)) \leqslant \Gamma_*(t). \tag{18}$$

b). Let $\rho_0 > \sigma_0$ and $0 < k < \mu$. Then the convergence function (13) increases up to the time η, where

$$k|z_0| = \int_0^\eta \gamma^*(v(s)) ds$$

and decreases for all $t > \eta$. Due to the view of function (10), it is clear to check that function $\gamma^*(v)$ is monotonically increasing with $\langle v, \xi_0 \rangle$. Consequently, for the convergence function (13), we may write estimates

$$\bar{\Gamma}_k(t, v(\cdot)) \leqslant 1 + kt - \frac{1}{|z_0|} \min_{v(\cdot)\in \mathbf{V}_I} \int_0^t (t-s)\gamma^*(v(s))ds \leqslant$$

$$\leqslant 1 + kt - \frac{1}{|z_0|} \int_0^t (t-s) \left(-\frac{\delta}{2} - |v(s)| + \sqrt{\left(-\frac{\delta}{2} - |v(s)|\right)^2 + b} \right) ds,$$

or

$$\bar{\Gamma}_k(t, v(\cdot)) \leqslant 1 + kt + \frac{1}{|z_0|} \int_0^t (t-s) \left(\frac{\delta}{2} + |v(s)| - \sqrt{\left(\frac{\delta}{2} + |v(s)|\right)^2 + b} \right) ds. \tag{19}$$

By introducing the new notation, which is analogous to that in (14), for the functions within the integral in (19) and examining these functions, it is not difficult to see that the Jensen inequality (15) and the Cauchy — Schwartz's inequality (17) also hold in this case. Therefore, we find the following results:

$$\bar{\Gamma}_k(t, v(\cdot)) \leqslant 1 + kt + \frac{t}{|z_0|} \left[\frac{\delta}{4}t + \frac{\sigma_0}{\sqrt{2}} - \sqrt{\left(\frac{\delta}{4}t + \frac{\sigma_0}{\sqrt{2}}\right)^2 + \frac{b}{4}t^2} \right],$$

or

$$\bar{\Gamma}_k(t, v(\cdot)) \leqslant \Gamma^*(t). \tag{20}$$

Based on Proposition 1, the right-hand sides of inequalities (18) and (20) vanish at the time $t = T_k$. Hence, we can express

$$0 \leqslant \bar{\Gamma}_k(t, v(\cdot)) \leqslant \Gamma_k(t)$$

on the time interval $[0, T_k]$ and this completes the proof. □

Theorem 1. *Let Lemma 1 be met. Then the pursuer wins by employing the* Π*-strategy* (11) *during the time interval* $[0, T_k]$*, where* T_k *is defined by Proposition 1.*

P r o o f. Let the evader choose an arbitrary control $v(\cdot)$, $v(\cdot) \in \mathbf{V}_I$, and let the pursuer employ Π-strategy (11). Then, by virtue of (7), we obtain the function

$$z(t) = z_0 + kz_0t - \int_0^t (t-s)\gamma_k(v(s))\xi_0 ds,$$

or we can write it down as result

$$z(t) = z_0\bar{\Gamma}_k(t, v(\cdot)). \tag{21}$$

Taking into account Lemma 2, $\bar{\Gamma}_k(t, v(\cdot)) \leqslant \Gamma_k(t)$ and $\Gamma_k(T_k) = 0$, there exists time $\theta \in [0, T_k]$ depending on $v(\cdot) \in \mathbf{V}_I$ such that $\bar{\Gamma}_k(\theta, v(\cdot)) = 0$ for all $\rho_0 \geqslant \sigma_0$ and $k \in (-\infty, \mu)$. Thus, by virtue of (21) the desired result $z(\theta) = 0$, i.e., $x(\theta) = y(\theta)$ is obtained.

It remains only to check that Π-strategy (11) is admissible for each $t \in [0, \theta]$. By (13) $\bar{\Gamma}_k(\theta, v(\cdot)) = 0$ implies that

$$\int_0^\theta (\theta - s)\gamma_k(v(s))ds = |z_0|(1 + k\theta). \tag{22}$$

Due to (4), (12), (22) and assuming $\theta \in [0, T_k]$, for $\rho_0 \geqslant \sigma_0$ and $k < 0$, we can write

$$\int_0^\theta (\theta - s)|\boldsymbol{u}_k(v(s))|^2 ds = \int_0^\theta (\theta - s)|v(s)|^2 ds + \delta \int_0^\theta (\theta - s)\gamma_k(v(s))ds + a \int_0^\theta (\theta - s)ds \leqslant$$

$$\leqslant \sigma_0^2 + \delta|z_0|(1 + k\theta) - \frac{\theta^2}{T_k} k\delta|z_0| \leqslant \rho_0^2.$$

In the same manner, for $\rho_0 > \sigma_0$ and $0 < k < \mu$, we can write

$$\int_0^\theta (\theta - s)|\boldsymbol{u}_k(v(s))|^2 ds = \int_0^\theta (\theta - s)|v(s)|^2 ds - \delta \int_0^\theta (\theta - s)\gamma_k(v(s))ds + b \int_0^\theta (\theta - s)ds \leqslant$$

$$\leqslant \sigma_0^2 - \delta|z_0|(1 + k\theta) + \frac{\theta^2 \delta|z_0|}{T_k}\left(k + \frac{2}{T_k}\right) \leqslant 2\sigma_0^2 - \rho_0^2 + 2\delta|z_0| = \rho_0^2.$$

Thus, strategy (11) is admissible. □

3.2. Evasion problem. In this subsection, the evasion problem will be considered. To address this problem, it is necessary to introduce the concept of the evader's strategy. In this case, as with the pursuer's strategy, it cannot be defined as a map $\mathbf{U}_I \to \mathbf{V}_I$ that satisfies the Volterra property, as this leads to an encounter with a vicious circle. If the control vector of the evader satisfy geometric constraints, its strategy can be chosen in the form of a constant vector as games of considering type [16, 18]. However, in the case of integral constraints being considered in our work, the constant strategy is not viable. To overcome this obstacle, we use L. A. Petrosyan's concept of an information-delayed strategy with the appropriate modification (see [5, p. 169]). In this context, we choose a simpler approach by constructing a concrete strategy for the evader. This strategy is defined so that, at each current time t, the pursuer's control values are accepted with a delay of the amount τ, where τ is a sufficiently small positive number, that is, vicious circle won't occur. For this purpose, we take the function $\widetilde{\mathbf{v}}(u) = -|u|\xi_0$ ($\xi_0 = z_0/|z_0|$) as the evader's strategy. Now, let the pursuer select any optional control $u(\cdot) \in \mathbf{U}_I$. Then the function that acts as described below

$$\mathbf{v}_\tau(t) = \begin{cases} 0, & \text{if } \ 0 \leqslant t < \tau, \\ \widetilde{\mathbf{v}}[u(t - \tau)], & \text{if } \ t \geqslant \tau, \end{cases} \tag{23}$$

is determined.

It can be said that the evader's strategy uses only the values of the control $u(\cdot)$ of the pursuer in the interval $[0, t - \tau]$ for each current t, $t \geqslant \tau$. Unlike in [5], in our case, the strategy operates continuously, not over the intervals $[0, \tau)$, $[\tau, 2\tau)$, $[2\tau, 3\tau)$. Obviously, if the initial values are given z_0, z_1, then the corresponding trajectory $z(t)$ is to be defined by the formulas

$$z(t) = z_0 + kz_0t + \int\limits_0^t (t-s)u(s)ds, \quad \text{if } \ 0 \leqslant t < \tau, \tag{24}$$

$$z(t) = z_0 + kz_0t + \int\limits_0^t (t-s)u(s)ds - \int\limits_\tau^t (t-s)\widetilde{\mathbf{v}}[u(s-\tau)]ds, \quad \text{if } \ t \geqslant \tau. \tag{25}$$

Theorem 2. *Let these conditions* $\rho_0 \leqslant \sigma_0$, $0 < \tau < \sqrt{2}|z_0|/\rho_0$ *and* $k \geqslant \rho_0/\sqrt{2}|z_0|$ *be met. Then the evader wins by using the strategy* (23) *in the time interval* $[0, +\infty)$.

P r o o f. Suppose that the pursuer begins its motion with a control $u(\cdot) \in \mathbf{U}_I$, and let the evader make use of the control (23). In view of (24), we have

$$|z(t)| \geqslant \big|z_0(1+kt)\big| - \int\limits_0^t (t-s)|u(s)|ds. \tag{26}$$

To commence, we will prove that evasion is feasible for every $t \in [0, \tau)$. The ensuing calculations, derived from the Cauchy—Schwartz inequality, the integral constraint (2), and the second condition of Theorem 2, support this

$$\int\limits_0^t (t-s)|u(s)|ds \leqslant \sqrt{\int\limits_0^t (t-s)ds}\sqrt{\int\limits_0^t (t-s)|u(s)|^2ds} \leqslant \frac{t\rho_0}{\sqrt{2}} \leqslant \frac{\tau\rho_0}{\sqrt{2}} < |z_0|.$$

We can infer from this and from (26) that $|z(t)| > |z_0|kt > 0$.

Next, we will now demonstrate the possibility of evasion throughout the time interval $[\tau, +\infty)$. In view of (25), we have formula

$$|z(t)| \geqslant \Big|z_0(1+kt) - \int\limits_\tau^t (t-s)\widetilde{\mathbf{v}}[u(s-\tau)]ds\Big| - \int\limits_0^t (t-s)|u(s)|ds. \tag{27}$$

By leveraging inequality (27), we can see that

$$|z(t)| \geqslant |z_0|(1+kt) + \int\limits_\tau^t (t-s)|u(s-\tau)|ds - \int\limits_0^t (t-s)|u(s)|ds =$$

$$= |z_0|(1+kt) + \int\limits_0^{t-\tau} (t-s-\tau)|u(s)|ds - \int\limits_0^{t-\tau} (t-s)|u(s)|ds - \int\limits_{t-\tau}^t (t-s)|u(s)|ds,$$

or in conclusion,

$$|z(t)| \geqslant |z_0|(1+kt) - \tau\int\limits_0^{t-\tau} |u(s)|ds - \int\limits_{t-\tau}^t (t-s)|u(s)|ds. \tag{28}$$

Applying the Cauchy — Schwartz inequality to the integrals on the right-hand side of (28), taking into consideration inequality (2), and, from the condition $0 < \tau < \sqrt{2}|z_0|/\rho_0$ we yield the following relations:

$$\int\limits_0^{t-\tau} |u(s)|ds \leqslant \sqrt{\int\limits_0^{t-\tau} \frac{1}{t-s}ds}\sqrt{\int\limits_0^{t-\tau} (t-s)|u(s)|^2 ds} \leqslant \rho_0\sqrt{\ln\frac{t}{\tau}},$$

and

$$\int\limits_{t-\tau}^{t} (t-s)|u(s)|ds \leqslant \frac{\tau\rho_0}{\sqrt{2}} < |z_0|.$$

Consequently, inequality (28) assumes the form

$$|z(t)| > \frac{\rho_0\tau}{\sqrt{2}}\left(\frac{\sqrt{2}|z_0|k}{\rho_0}\cdot\frac{t}{\tau} - \sqrt{\ln\frac{t^2}{\tau^2}}\right). \tag{29}$$

According to $k \geqslant \rho_0/\sqrt{2}|z_0|$, inequality (29) guarantees that $|z(t)| > 0$ for every $t \geqslant \tau$.

To complete the proof, let's make sure the admissibility of the strategy (23) in the time interval $[\tau, +\infty)$ (it is clear to see in the time interval $[0, \tau)$). If $u(\cdot) \in \mathbf{U}_I$ is an optional control of the pursuer, then according to the first condition of Theorem 2, it is derived from (4) that

$$\int\limits_0^t (t-s)|\mathbf{v}_\tau(s)|^2 ds = \int\limits_\tau^t (t-s)|u(s-\tau)|^2 ds = \int\limits_0^{t-\tau} (t-\tau-s)|u(s)|^2 ds \leqslant \rho_0^2 \leqslant \sigma_0^2. \quad \square$$

4. The differential game with "life line" problem.

4.1. Pursuit problem in the case $z_1 = 0$.

Definition 7. In the differential game (1)–(4), the function

$$\boldsymbol{u}_0(v) = v - \gamma_0(v)\xi_0 \tag{30}$$

is called the Π-strategy of the pursuer for the case $z_1 = 0$, where

$$\gamma_0(v) = \max\{0, \delta + 2\langle v, \xi_0\rangle\}, \quad \xi_0 = z_0/|z_0|, \quad \delta = (\rho_0^2 - \sigma_0^2)/|z_0|.$$

In line with the previous discussions in [18], let us consider the following assertion for the strategy (30) and the resolving function $\gamma_0(v)$.

Lemma 3. *If $\delta \geqslant 0$ holds, then the function $\gamma_0(v)$ is well-defined and nonnegative for any control $v(\cdot) \in \mathbf{V}_I$, and, what's more, the equality*

$$|\boldsymbol{u}_0(v)|^2 = |v|^2 + \delta\gamma_0(v), \tag{31}$$

is met.

Suppose that the evader picks some admissible control $v(\cdot) \in \mathbf{V}_I$, and in response, the pursuer adopts the function $\boldsymbol{u}_0(v(t))$. By virtue of the equations (1) and (2), we can determine the trajectories of both the pursuer and evader:

$$x(t) = x_0 + x_1 t + \int\limits_0^t (t-s)\boldsymbol{u}_0(v(s))ds, \tag{32}$$

$$y(t) = y_0 + y_1 t + \int_0^t (t-s)v(s)ds, \tag{33}$$

and their residual resources

$$\rho(t) = \rho_0^2 - \int_0^t (t-s)|\boldsymbol{u}_0(v(s))|^2 ds, \quad \rho(0) = \rho_0^2, \tag{34}$$

$$\sigma(t) = \sigma_0^2 - \int_0^t (t-s)|v(s)|^2 ds, \qquad \sigma(0) = \sigma_0^2, \tag{35}$$

respectively.

Let us examine the scalar function

$$\bar{\Gamma}_0(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \int_0^t (t-s)\gamma_0(v(s))ds \tag{36}$$

with respect to t, $t \geqslant 0$.

Let $\rho_0 > \sigma_0$. Then scalar function (36) is called the convergence function of the players in the pursuit game for the case $z_1 = 0$ and for any $v(\cdot) \in \mathbf{V}_I$, this function is monotonically decreasing in t, $t \geqslant 0$. We can express the subsequent approximations for this convergence function:

$$\bar{\Gamma}_0(t, v(\cdot)) \leqslant 1 - \frac{1}{|z_0|} \min_{v(\cdot)\in\mathbf{V}_I} \int_0^t (t-s)\max\{0, \delta + 2\langle v(s), \xi_0\rangle\}\, ds \leqslant$$

$$\leqslant 1 - \frac{1}{|z_0|} \max\left\{0, \delta\int_0^t (t-s)ds - 2\int_0^t (t-s)|v(s)|ds\right\}.$$

Applying the Cauchy—Schwartz inequality (17) to the integral in the last expression and in light of constraint (4), we find that the function $\bar{\Gamma}_0(t, v(\cdot))$ is bounded from above by the function $\Gamma_0(t)$, i.e.,

$$\bar{\Gamma}_0(t, v(\cdot)) \leqslant \Gamma_0(t), \tag{37}$$

here $\Gamma_0(t) = 1 - 1/|z_0| \max\left\{0, \delta t^2/2 - \sqrt{2}\sigma_0 t\right\}$.

Theorem 3. *If $\rho_0 > \sigma_0$, then for the case $z_1 = 0$, the pursuer wins by using* Π*-strategy* (30) *on the time interval* $[0, T_0]$, *where* $T_0 = \sqrt{2}|z_0|/(\rho_0 - \sigma_0)$.

P r o o f. We previously established that the trajectories (32), (33) are generated by an arbitrary control $v(\cdot) \in \mathbf{V}_I$ of the evader and Π-strategy (30) of the pursuer, and thus, because of (7), for the case $z_1 = 0$ $(k = 0)$, we find $z(t) = z_0 - \int_0^t (t-s)\gamma_0(v(s))\xi_0 ds$, alternatively, we can record it in the following format:

$$z(t) = z_0\bar{\Gamma}_0(t, v(\cdot)). \tag{38}$$

Taking into account the inequality (37), it can be easily verified that $\Gamma_0(T_0) = 0$. So, we find $\theta_0 \in [0, T_0]$ depending on $v(\cdot) \in \mathbf{V}_I$ such that $\bar{\Gamma}(\theta_0, v(\cdot)) = 0$. Hence, by virtue of (38) the desired result $z(\theta_0) = 0$, i.e., $x(\theta_0) = y(\theta_0)$ is obtained.

To complete the proof, we must verify the admissibility of Π-strategy (30) for each $t \in [0, \theta_0]$. From (31)

$$\int_0^{\theta_0} (\theta_0 - s)|\boldsymbol{u}_0(v(s))|^2 ds = \int_0^{\theta_0} (\theta_0 - s)|v(s)|^2 ds + \delta \int_0^{\theta_0} (\theta_0 - s)\gamma_0(v(s))ds \leqslant \sigma_0^2 + \delta|z_0| = \rho_0^2.$$

□

For $\rho_0 > \sigma_0$, the pursuer applies Π-strategy (30). Then, for all $t \in [0, \theta_0]$ and for the functions (34), (35), by (31) we form

$$\rho(t) - \sigma(t) = \rho_0^2 - \sigma_0^2 - \delta \int_0^t (t - s)\gamma_0(v(s))ds = \delta|z_0|\bar{\Gamma}_0(t, v(\cdot)). \tag{39}$$

4.2. A meeting domain of the players. The current subsection is devoted to investigating the dynamics of the meeting domain of the players in the pursuit game for the case $z_1 = 0$ $(x_1 = y_1)$ and the "life-line" game problem of R. Isaacs.

Let the quadruplets (x, y, ρ, σ) describe the current state of the game at some moment in time t, where $x \neq y$. The set $\Omega\big(x, y, \rho, \sigma\big)$ is defined as the set of all points ω where the pursuer moving from the position x and consuming the resource ρ should encounter the evader moving from the position y and consuming the resource σ, i.e.

$$\Omega\big(x, y, \rho, \sigma\big) = \Big\{\omega : |\omega - x| \geqslant \sqrt{\rho/\sigma}|\omega - y|\Big\}. \tag{40}$$

For $\rho \neq \sigma$, the boundary of set (40) is $\Omega_b\,(x, y, \rho, \sigma) = \Big\{\omega : |\omega - x| = \sqrt{\rho/\sigma}\,|\omega - y|\Big\}$, which is known as Apollonius' sphere. The center and radius of this sphere are

$$C(x, y, \rho, \sigma) = (\rho y - \sigma x)/(\rho - \sigma), \quad R(x, y, \rho, \sigma) = \sqrt{\rho\sigma}|x - y|/(\rho - \sigma).$$

We mentioned that for any control $v(\cdot) \in \mathbf{V}_I$ and for the Π-strategy (30), the triplets $(y_0, y_1, v(\cdot))$ and $(x_0, x_1, \boldsymbol{u}_0(v(\cdot)))$, $\boldsymbol{u}_0(v(\cdot)) \in \mathbf{U}_I$ generate the trajectories of the players (32) and (33) while $t \in [0, \theta_0]$, $0 \leqslant \theta_0 \leqslant T_0$, where θ_0 is the encounter time of the players, which the equality $x(\theta_0) = y(\theta_0)$ holds. Accordingly, by means of (40), for each quadruplet $(x(t), y(t), \rho(t), \sigma(t))$, $t \in [0, \theta_0]$, we define the following multi-valued mapping:

$$\Omega\big(x(t), y(t), \rho(t), \sigma(t)\big) = \Big\{\omega : \big|\omega - x(t)\big| \geqslant \sqrt{\rho(t)/\sigma(t)}\big|\omega - y(t)\big|\Big\},$$

or

$$\Omega\big(x(t), y(t), \rho(t), \sigma(t)\big) = C\big(x(t), y(t), \rho(t), \sigma(t)\big) + R\big(x(t), y(t), \rho(t), \sigma(t)\big)S, \tag{41}$$

as long as $\sigma(t) > 0$ on $[0, \theta_0]$.

On $[0, \theta_0]$, it is obvious that the inclusion $y(t) \in \Omega\big(x(t), y(t), \rho(t), \sigma(t)\big)$ is valid. From (38) and (39), the view of the multi-valued mapping $\Omega\big(x(t), y(t), \rho(t), \sigma(t)\big)$ (41) can be expressed as

$$\Omega\big(x(t), y(t), \rho(t), \sigma(t)\big) = x(t) + \frac{\sqrt{\rho(t)\sigma(t)}}{\delta}S - \frac{\rho(t)}{\delta}\xi_0, \quad t \in [0, \theta_0], \tag{42}$$

here S is the unit ball whose centre is at the origin of the space $\mathbb{R}^n$.

Lemma 4 (Petrosyan type lemma [5]**).** *The multi-valued mapping* $\Omega(t) - tx_1$, $t \in [0, \theta_0]$, *is monotonically decreasing with respect to the inclusion, i.e. if* t_1, $t_2 \in [0, \theta_0]$ *and* $t_1 < t_2$, *then*

$$\Omega(t_1) - t_1 x_1 \supset \Omega(t_2) - t_2 x_1,$$

where $\Omega(t) = \Omega\big(x(t), y(t), \rho(t), \sigma(t)\big)$.

P r o o f. Let $a(t) = \rho(t)/\sigma(t)$, and let us start with the inequality

$$\left[\left(\sqrt{a(t)} - \sqrt{\frac{1}{a(t)}}\right) \int\limits_0^t |v(s)|^2 ds + \delta\sqrt{a(t)} \int\limits_0^t \gamma_0(v(s)) ds\right]^2 \geqslant 0.$$

Add the quantity $4\Big(\int_0^t |v(s)|^2 ds\Big)^2 + 4\delta \int_0^t |v(s)|^2 ds \int_0^t \gamma_0(v(s)) ds$ to both sides of this inequality, from the view of function $\gamma_0(v)$ and according to the Cauchy—Schwartz inequality, then we find

$$\left[\left(\sqrt{a(t)} + \sqrt{\frac{1}{a(t)}}\right) \int\limits_0^t |v(s)|^2 ds + \delta\sqrt{a(t)} \int\limits_0^t \gamma_0(v(s)) ds\right]^2 \geqslant$$

$$\geqslant 4\left[\left(\int\limits_0^t |v(s)|^2 ds\right)^2 + \delta \int\limits_0^t |v(s)|^2 ds \int\limits_0^t \gamma_0(v(s)) ds\right] \geqslant 4\left|\xi_0 \int\limits_0^t |v(s)|^2 ds + \delta \int\limits_0^t v(s) ds\right|^2.$$

As a result, we can formulate the inequality

$$\left|\left(\sqrt{a(t)} + \sqrt{\frac{1}{a(t)}}\right) \int\limits_0^t |v(s)|^2 ds + \delta\sqrt{a(t)} \int\limits_0^t \gamma_0(v(s)) ds\right| \geqslant$$

$$\geqslant 2\left\langle \xi_0 \int\limits_0^t |v(s)|^2 ds + \delta \int\limits_0^t v(s) ds, \psi \right\rangle,$$

for all $\psi \in \mathbb{R}^n$, $|\psi| = 1$. Due to (31), it is easy to calculate that

$$-2\frac{d}{dt}\sqrt{\rho(t)\sigma(t)} = \left(\sqrt{\frac{\rho(t)}{\sigma(t)}} + \sqrt{\frac{\rho(t)}{\sigma(t)}}\right) \int\limits_0^t |v(s)|^2 ds + \delta\sqrt{\frac{\rho(t)}{\sigma(t)}} \int\limits_0^t \gamma_0(v(s)) ds =$$

$$= \left(\sqrt{a(t)} + \sqrt{\frac{1}{a(t)}}\right) \int\limits_0^t |v(s)|^2 ds + \delta\sqrt{a(t)} \int\limits_0^t \gamma_0(v(s)) ds.$$

From here, we obtain

$$-\frac{d}{dt}\sqrt{\rho(t)\sigma(t)} \geqslant \left\langle \xi_0 \int\limits_0^t |v(s)|^2 ds + \delta \int\limits_0^t v(s) ds, \psi \right\rangle, \tag{43}$$

for all $\psi \in \mathbb{R}^n$, $|\psi| = 1$. By Π-strategy (30) and from (31), we have

$$\left\langle \xi_0 \int_0^t |v(s)|^2 ds + \delta \int_0^t v(s) ds, \psi \right\rangle =$$

$$= \big\langle \xi_0, \psi \big\rangle \int_0^t |\boldsymbol{u}_0(v(s))|^2 ds + \delta \Big\langle \int_0^t \boldsymbol{u}_0(v(s)) ds, \psi \Big\rangle = -\langle \xi_0, \psi \rangle \frac{d}{dt}\rho(t) + \delta \Big\langle \int_0^t \boldsymbol{u}_0(v(s)) ds, \psi \Big\rangle,$$

or, taking into account (43), we find

$$-\frac{d}{dt}\sqrt{\rho(t)\sigma(t)} \geqslant -\langle \xi_0, \psi \rangle \frac{d}{dt}\rho(t) + \delta \Big\langle \int_0^t \boldsymbol{u}_0(v(s)) ds, \psi \Big\rangle. \tag{44}$$

For arbitrary $\psi \in \mathbb{R}^n$, $|\psi| = 1$, the multi-valued mapping $\Omega(x(t), y(t), \rho(t), \sigma(t)) - tx_1$ has a support function (see [24])

$$F\Big(\Omega\big(x(t), y(t), \rho(t), \sigma(t)\big) - tx_1, \psi\Big) = \sup_{\omega \in \Omega\left(x(t), y(t), \rho(t), \sigma(t)\right) - tx_1} \big\langle \omega, \psi \big\rangle.$$

On the strength of the properties of the support function, we use (32), (42) to calculate the derivative of $F\left(\Omega\big(x(t), y(t), \rho(t), \sigma(t)\big) - tx_1, \psi\right)$ with t, i.e.,

$$\frac{d}{dt}F\Big(\Omega\big(x(t), y(t), \rho(t), \sigma(t)\big) - tx_1, \psi\Big) = \frac{d}{dt}F\Big(\Omega\big(x(t), y(t), \rho(t), \sigma(t)\big), \psi\Big) - \langle x_1, \psi \rangle =$$

$$= \frac{d}{dt}F\left(x_0 + x_1 t + \int_0^t (t-s)\boldsymbol{u}_0(v(s)) ds + \frac{\sqrt{\rho(t)\sigma(t)}}{\delta} S - \frac{\rho(t)}{\delta}\xi_0, \psi\right) - \langle x_1, \psi \rangle =$$

$$= \Big\langle \int_0^t \boldsymbol{u}_0(v(s)) ds, \psi \Big\rangle + \frac{1}{\delta}\langle S, \psi \rangle \frac{d}{dt}\sqrt{\rho(t)\sigma(t)} - \frac{1}{\delta}\langle \xi_0, \psi \rangle \frac{d}{dt}\rho(t).$$

Considering $\langle S, \psi \rangle = 1$ (see [24, p. 33]), from here and from (44), we derive the following results:

$$\frac{d}{dt}F\Big(\Omega\big(x(t), y(t), \rho(t), \sigma(t)\big) - tx_1, \psi\Big) =$$

$$= \frac{1}{\delta}\left[\delta \Big\langle \int_0^t \boldsymbol{u}_0(v(s)) ds, \psi \Big\rangle + \frac{d}{dt}\sqrt{\rho(t)\sigma(t)} - \langle \xi_0, \psi \rangle \frac{d}{dt}\rho(t)\right] \leqslant 0,$$

for any $\psi \in \mathbb{R}^n$, $|\psi| = 1$. This finishes the proof of Lemma 4. $\square$

Corollary. From Lemma 4, it can be concluded that:

a) $\Omega\big(x(t), y(t), \rho(t), \sigma(t)\big) \subset \Omega\big(x_0, y_0, \rho_0, \sigma_0\big) + tx_1$ at each $t \in [0, \theta_0]$;

b) $y(t) \in \Omega\big(x_0, y_0, \rho_0, \sigma_0\big) + tx_1$ for all $t \in [0, \theta_0]$ is met, where

$$\Omega\big(x_0, y_0, \rho_0, \sigma_0\big) = x_0 + \frac{\rho_0 \sigma_0}{\delta} S - \frac{\rho_0^2}{\delta}\xi_0.$$

Definition 8. The set

$$\Omega^*\big(x_0, y_0, \rho_0, \sigma_0, T_0\big) = \bigcup_{t=0}^{T_0} \big\{\Omega\big(x_0, y_0, \rho_0, \sigma_0\big) + tx_1\big\}$$

is called the meeting domain of the players in the pursuit game for the case $x_1 = y_1$, where T_0 is the guaranteed capture time.

Let W be the closed subset of $\mathbb{R}^n$, and let for the case $x_1 = y_1$ the differential game (1)–(4) with the "life-line" W be studied.

Definition 9. We say that the Π-strategy (30) provides winning for the pursuer in the "life-line" game on the time interval $[0, T_0]$, if there exists such a time moment $\theta_0 \in [0, T_0]$ that the conditions

a) $x(\theta_0) = y(\theta_0)$;

b) $y(t) \notin W$ for all $t \in [0, \theta_0]$ are valid.

Theorem 4. *Let $\rho_0 > \sigma_0$ and $\Omega^*\big(x_0, y_0, \rho_0, \sigma_0, T_0\big) \cap W = \emptyset$ for the case $x_1 = y_1$. Then the pursuer wins by using Π-strategy (30) on the time interval $[0, T_0]$ in the "life-line" game W, where $T_0 = \sqrt{2}|z_0|/(\rho_0 - \sigma_0)$.*

P r o o f. The proof immediately follows from Theorem 3, Lemma 4, Corollary. □

5. Example of meeting domain in the case of multi-pursuer and one evader. Let's take a look at the multi-player game below:

$$\ddot{x}_i = u_i, \quad x_i(0) = x_{i0}, \quad \dot{x}_i(0) = \vartheta, \quad \int_0^t (t-s)|u(s)|^2 ds \leqslant \nu_i,$$

$$\ddot{y} = v, \qquad y(0) = y_0, \quad \dot{y}(0) = \vartheta, \quad \int_0^t (t-s)|v(s)|^2 ds \leqslant 1,$$

where $\nu_i > 1$, $k = 0$, $x_{i0} \neq y_0$, $i = \overline{1, m}$, $\delta_i = (\nu_i - 1)/|z_{i0}|$. Considering Lemma 4 and Corollary, we write the relationship

$$y(t) \in \bigcap_{i=1}^{m} \Omega_i(x_{i0}, y_0) + t\vartheta,$$

here

$$\Omega_i(x_{i0}, y_0) = x_{i0} + \sqrt{\nu_i} S/\delta_i + \nu_i \xi_{i0}/\delta_i, \quad \xi_{i0} = z_{i0}/|z_{i0}|,$$

and S is the unit ball centered at the origin in $\mathbb{R}^2$. We can define the players' meeting domain in the following way:

$$\Omega^*_{P_1,\ldots,P_m}(x_{i0}, y_0, \vartheta, \widetilde{T}_0) = \bigcup_{t=0}^{\widetilde{T}_0} \left[\bigcap_{i=1}^{m} \Omega_i(x_{i0}, y_0) + t\vartheta\right],$$

where $\widetilde{T}_0 = \min_{i=\overline{1,m}} \left\{\sqrt{2}|z_{i0}|/(\sqrt{\nu_i} - 1)\right\}$.

6. Conclusion. In this paper, we have discussed the pursuit-evasion problems in a differential game with inertial players under integral constraints on controls. Pursuit problems have been solved by separating them into two cases. In the first case, there was a linear dependence of the difference between the players' initial state vectors and their initial velocity vectors, while in the second case, the initial velocities were the same. The key method employed to achieve the pursuit is the Π-strategy. Furthermore, the meeting domain of the players is constructed by applying the Π-strategy for the case, where the players have the identical initial speeds and we have considered the “life-line” game in favor of the pursuer for this case.

When solving the evasion problem, a specific strategy for the evader has been proposed with delayed information, and sufficient conditions for its solvability have been obtained. In the future, we can generalize the proposed method to more general linear and nonlinear conflict-controlled systems with various control constraints.

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Дифференциальная игра с инерционными игроками при интегральных ограничениях на управления

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Рассматриваются дифференциальные игры преследования-убегания при интегральных ограничениях с инерционными игроками (преследователь и убегающий). В случае совпадения позиции игроков считается, что убегающий захвачен преследователем. Основной метод, используемый для преследователя в дифференциальной игре преследования, — это стратегия параллельного преследования (кратко, П-стратегия). Получены достаточные условия для разрешимости задач преследования-убегания. Кроме того, исследуется игра «линия жизни» Айзекса в пользу преследователя в случае одинаковых начальных скоростей игроков, в которой аналитическая формула для области встречи игроков дана основной леммой, характеризующей ее свойство монотонности. Проведенная работа расширяет и продолжает труды Р. Айзекса, Л. А. Петросяна, Б. Н. Пшеничного, Н. Ю. Сатимова, авторов этой статьи и других исследователей.

Ключевые слова: дифференциальная игра, преследователь, убегающий, стратегия, игра «линия жизни».

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