ПРОЦЕССЫ УПРАВЛЕНИЯ

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Competition for agents' opinions in small dynamic systems with limited control*

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This paper explores a competitive opinion dynamic system representing a social network evolving over time. The agents compare their own opinions with the average ones and form new opinions. There are players who control agents' opinions in a social network, and their aims are to make opinions in the society closer to the desired levels over a finite time horizon by minimizing their costs. The feature of this model is that players can influence agents in a limited number of time moments. We show how players can choose the moments of influence from Pareto optimal set on numerical examples. The results of numerical simulations are provided in the work.

Keywords: opinion competition game, Nash equilibrium, opinion dynamics, social network.

1. Introduction. Opinion dynamics modeling can play an important role in addressing the issue of information dissemination in social networks. Within the existing literature, numerous models have been proposed to capture opinion dynamics, such as the classical DeGroot model [1], from which a number of variants have been derived, including the Friedkin — Johnsen model [2] and the bounded confidence model [3]. There are several studies considering the influence of average opinion in the society on individuals' opinions with limited observation capabilities in a linear quadratic optimal control problem [4–7].

This research addresses the multifaceted theme of competitive opinion dynamics, a phenomenon of increasing prominence in diverse social scenarios, such as online marketing, advertising, promotions, voting, etc. [8–10]. The overarching objective is to delve into the intricate mechanisms that underlie competition and opinion diffusion, which are pivotal factors in shaping opinions within connected communities.

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A scenario in which two centers of influence compete for the agents' attention within a network is considered in [11]. The analysis is limited with constraints imposed by a network structure. They find the necessary conditions of the Nash equilibrium and steady state for a given state dynamics. In another aspect, where SI_1SI_2S model is introduced to control dissemination of two opinions [12]. This model presents open-loop Nash control strategies that empower campaigners to actively monitor opinion spread and adapt their strategies in response. A innovative perspective is offered through the introduction of the cost-effective competition (CEC) problem [13].

A multi-objective optimization approach is developed to aim at achieving more votes with minimized recruitment costs. Unlike the DeGroot model, this research [14] introduces an innovative dimension accounting for both individual competition and switching topologies within a social network. The analysis reveals whether the structure of the network topology (balanced or not) affects opinions to reach consensus. The paper [15] examines a problem of influence maximization in a social network where two players compete by means of dynamic targeting strategies. The authors obtained some elements for the characterization of equilibrium strategies through model analysis. A game-theoretic model for competitive information dissemination in social network is proposed in [16]. It is shown that the speed of information spreading is influenced by characteristics of individuals. The authors [17] investigate the idea of keeping a scalar opinion of every agent above a predetermined ferment level over a finite time horizon. They obtain the optimal control trajectory with the turnpike property by using the Pontryagin maximum principle.

Influencers or media centers use various methods to control opinions of the social network members on the given topic, and then they try to keep the opinions closer to the desired level, this process can be modeled as a dynamic game of competition for agents' opinions [18–21].

In this paper, we consider a small social network consisting of two agents and two players, and each player can directly influence the opinion of a unique agent. We assume that players have a limited capability to access the agents and can influence their opinions in a limited number of time moments. Therefore, a player is interested in influencing the agent at the "right time", but he can observe the agents' opinions all the time. Players or competitors have desired opinions and their goal is to minimize the costs on influence keeping the agents' opinions closer to the desired ones in a given time horizon. We formulate a problem as a linear-quadratic dynamic game at discrete time with finite horizon and find the Nash equilibrium in open-loop strategies [22]. The Nash equilibrium is found for any given set of time moments when players control the agents. We discuss the result if the players have an option to choose these sets. The "right time" of influence corresponds to the least costs among all equilibrium costs with all possible time sets. In the numerical simulations we find Pareto-optimal pairs of players' costs.

The paper is organized as follows. Section 2 describes the game and provides the main theorem and its proof. In Section 3 the results of numerical simulations are given. Section 4 concludes the paper.

2. Competition game for agents' opinions with two players. We propose the following model: in a social network, the opinions of agents are represented by $x_i(t)$ at time t, where i is the number of an agent. Suppose there are two players who directly influence opinions of agents 1 and 2, respectively, and the level of influence is denoted by $u_j(t)$, j is the number of a player. The sets V_1, V_2, here $V_j = \left\{t_1^j, \ldots, t_k^j\right\}$, $j = 1, 2$, are the sets of time moments, in which players control the opinions of agents, and the number of elements k of set V_j is given. We assume that it is the same for both players. Define a two-player game of competition for agents' opinions with the set of players' strategies U_1, U_2, where $U_j = (u_j(t) \in \mathbb{R} \mid t \in V_j),\ j = 1, 2$. The players have the same discount factor, but their levels of influence per unit cost and target opinions are different. Summarize the notations:

• $\mathrm{x}_i(t),\ i = 1, 2$: the opinion of agent i at time $t \in \{0, 1, \ldots, T\}$;

• $\mathrm{u}_j(t),\ j = 1, 2$: player 1 influences agent 1's opinion with $u_1(t)$, $t \in V_1$, player 2 influences agent 2's opinion with $u_2(t)$ at time $t \in V_2$;

• $\mathrm{V}_j = \left\{t_1^j, \ldots, t_k^j \mid 0 \leqslant t_1^j < t_2^j < \cdots < t_k^j \leqslant T - 1\right\}$, $j = 1, 2$: the set of time moments, when player j controls the corresponding agent's opinion;

• $\mathrm{U}_j = (u_j(t) \in R \mid t \in V_j)$, $j = 1, 2$: players' strategy sets of control variables. The small social network we examine is represented in Figure 1.

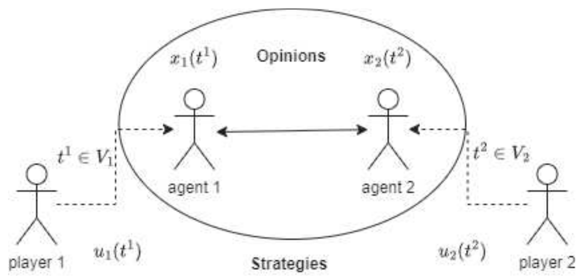

Figure 1. Small social network

The dynamics of agents' opinions are defined by the following equations:

$$x_1(t+1) = x_1(t) + a_1\left(\frac{x_1(t) + x_2(t)}{2} - x_1(t)\right) + u_1(t), \quad t \in V_1, \tag{1}$$

$$x_1(t+1) = x_1(t) + a_1\left(\frac{x_1(t) + x_2(t)}{2} - x_1(t)\right), \quad t \notin V_1, \tag{2}$$

$$x_2(t+1) = x_2(t) + a_2\left(\frac{x_1(t) + x_2(t)}{2} - x_2(t)\right) + u_2(t), \quad t \in V_2, \tag{3}$$

$$x_2(t+1) = x_2(t) + a_2\left(\frac{x_1(t) + x_2(t)}{2} - x_2(t)\right), \quad t \notin V_2, \tag{4}$$

with initial condition

$$x_1(0) = x_1^0, \quad x_2(0) = x_2^0.$$

In equations (1)–(4) $a_1 > 0$, $a_2 > 0$ denote agent 1 and 2's beliefs about the average social opinion, respectively.

The players' target opinions are s_1 and $s_2 \in \mathbb{R}$. Players 1 and 2 are willing to minimize functions:

$$J_1(u_1, u_2) = \sum_{t_i \in V_1} \delta^{t_i}\left(c_1 u_1^2(t_i)\right) + \sum_{t=0}^{T} \delta^t \left((x_1(t) - s_1)^2 + (x_2(t) - s_1)^2\right),$$

$$J_2(u_1, u_2) = \sum_{t_i \in V_2} \delta^{t_i}\left(c_2 u_2^2(t_i)\right) + \sum_{t=0}^{T} \delta^t \left((x_1(t) - s_2)^2 + (x_2(t) - s_2)^2\right),$$

where $\delta \in (0, 1]$ is a discount factor and $c_j > 0$ is player j's cost per unit level of influence.

Remark 1. The cost of influencing the agent can have a linear term, but in this study we simplified this part to simplify the analysis. There can be also fixed costs in the cost function, such as using the same strategy that aids the player's intervention regardless of when the player intervenes with the agent, but one does not know how influential it is and whether it can have a positive impact on promoting the agent's opinion closer to the target's opinion, so the fixed costs do not depend on the player's influence level.

We find the Nash equilibrium in open-loop strategies. The advantages of this type of strategies are simplicity and predictability. Open-loop strategies do not require continuous feedback based on the current state of the system and are therefore simpler to compute. These strategies are predefined and therefore easier to implement and predict. The largest disadvantage of open-loop strategies is their lack of adaptability: they cannot adapt to changes in the system state within the planning horizon. This can be a significant limitation in dynamic environments where the system state may change unpredictably.

Theorem. *Let* $\{(u_1^*, u_2^*), u_j = (u_j(t) : t \in V_j), j = 1, 2\}$ *be the Nash equilibrium in the game and* $\{(x_1^*(t), x_2^*(t)) : t = 0, \ldots, T\}$ *be a state trajectory corresponding to this equilibrium with initial condition* $x_1(0) = x_1^0$, $x_2(0) = x_2^0$, *then they satisfy the system*

$$\begin{cases} u_1(t) = \frac{\delta}{2c_1}\lambda_1^1(t+1), \ t \in V_1, \\ \lambda_1^1(t+1) = \frac{2}{\delta(2-a_1)}\left[2(x_1(t) - s_1) - \lambda_1^1(t) - \lambda_1^2(t+1)\frac{a_2\delta}{2}\right], \ t = 0, \ldots, T-1, \\ \lambda_1^2(t+1) = \frac{2}{\delta(2-a_2)}\left[2(x_2(t) - s_1) - \lambda_1^1(t+1)\frac{a_1\delta}{2} - \lambda_1^2(t)\right], \ t = 0, \ldots, T-1, \\ \lambda_1^1(T) = 2(x_1(T) - s_1), \\ \lambda_1^2(T) = 2(x_2(T) - s_1), \\ u_2(t) = \frac{\delta}{2c_2}\lambda_2^2(t+1), \ t \in V_2, \\ \lambda_2^1(t+1) = \frac{2}{\delta(2-a_1)}\left[2(x_1(t) - s_2) - \lambda_2^1(t) - \lambda_2^2(t+1)\frac{a_2\delta}{2}\right], \ t = 0, \ldots, T-1, \\ \lambda_2^2(t+1) = \frac{2}{\delta(2-a_2)}\left[2(x_2(t) - s_2) - \lambda_2^1(t+1)\frac{a_1\delta}{2} - \lambda_2^2(t)\right], \ t = 0, \ldots, T-1, \\ \lambda_2^1(T) = 2(x_1(T) - s_2), \\ \lambda_2^2(T) = 2(x_2(T) - s_2), \end{cases}$$

taking into account state equations (1)–(4) *and initial state* $(x_1(0),\ x_2(0)) = \left(x_1^0,\ x_2^0\right)$.

P r o o f. We find the strategy profile $(u_1^*,\ u_2^*)$, which is the Nash equilibrium in the game described above. We find the equilibrium in open-loop strategies using the Pontryagin maximum principle. The Hamiltonian of player 1 is

$$\begin{gathered} H_1^1\left(x_1(t), x_2(t), \lambda_1^1(t+1), \lambda_1^2(t+1), u_1(t), u_2(t), t\right) = \\ = c_1 u_1^2(t) + (x_1(t) - s_1)^2 + (x_2(t) - s_1)^2 + \\ + \delta\lambda_1^1(t+1)\left(x_1(t+1) - x_1(t) - a_1\left(\frac{x_1(t) + x_2(t)}{2} - x_1(t)\right) - u_1(t)\right) + \\ + \delta\lambda_1^2(t+1)\left(x_2(t+1) - x_2(t) - a_2\left(\frac{x_1(t) + x_2(t)}{2} - x_2(t)\right) - u_2(t)\right), \end{gathered}$$

for any $t \in V_1$, and it takes the form

$$\begin{gathered} H_1^2\left(x_1(t), x_2(t), \lambda_1^1(t+1), \lambda_1^2(t+1), t\right) = \\ = (x_1(t) - s_1)^2 + (x_2(t) - s_1)^2 + \\ + \delta\lambda_1^1(t+1)\left(x_1(t+1) - x_1(t) - a_1\left(\frac{x_1(t) + x_2(t)}{2} - x_1(t)\right)\right) + \\ + \delta\lambda_1^2(t+1)\left(x_2(t+1) - x_2(t) - a_2\left(\frac{x_1(t) + x_2(t)}{2} - x_2(t)\right)\right), \end{gathered}$$

for any $t \notin V_1$.

Finding the derivatives $\frac{\partial H_1^1(t)}{\partial u_1(t)} = 0,\ t \in V_1,\ \lambda_1^1(t) = \frac{\partial H_1^1(t)}{\partial x_1(t)} = \frac{\partial H_1^2(t)}{\partial x_1(t)}$ and $\lambda_1^2(t) = \frac{\partial H_1^1(t)}{\partial x_2(t)} = \frac{\partial H_1^2(t)}{\partial x_2(t)},\ t = 1, \dots, T-1$, we obtain the system of equations

$$\frac{\partial H_1^1(t)}{\partial u_1(t)} = 2c_1 u_1(t) - \delta\lambda_1^1(t+1) = 0,\ t \in V_1,$$
$$\lambda_1^1(t) = \frac{\partial H_1^1(t)}{\partial x_1(t)} = \frac{\partial H_1^2(t)}{\partial x_1(t)} =$$
$$= 2(x_1(t) - s_1) - \delta\lambda_1^1(t+1)\left(1 - \frac{a_1}{2}\right) - \delta\lambda_1^2(t+1)\frac{a_2}{2},$$
$$t = 1, \dots, T-1,$$
$$\lambda_1^2(t) = \frac{\partial H_1^1(t)}{\partial x_2(t)} = \frac{\partial H_1^2(t)}{\partial x_2(t)} =$$
$$= 2(x_2(t) - s_1) - \delta\lambda_1^1(t+1)\frac{a_1}{2} - \delta\lambda_1^2(t+1)\left(1 - \frac{a_2}{2}\right),$$
$$t = 1, \dots, T-1,$$
$$\lambda_1^1(T) = \frac{\partial\left((x_1(T) - s_1)^2 + (x_2(T) - s_1)^2\right)}{\partial x_1(T)} = 2(x_1(T) - s_1),$$
$$\lambda_1^2(T) = \frac{\partial\left((x_1(T) - s_1)^2 + (x_2(T) - s_1)^2\right)}{\partial x_2(T)} = 2(x_2(T) - s_1).$$

It can be rewritten as the system

$$\begin{cases} u_1(t) = \frac{\delta}{2c_1}\lambda_1^1(t+1),\ t \in V_1, \\ \lambda_1^1(t+1) = \frac{2}{\delta(2-a_1)}\left[2(x_1(t) - s_1) - \lambda_1^1(t) - \lambda_1^2(t+1)\frac{a_2\delta}{2}\right], \\ \quad t = 0, \dots, T-1, \\ \lambda_1^2(t+1) = \frac{2}{\delta(2-a_2)}\left[2(x_2(t) - s_1) - \lambda_1^1(t+1)\frac{a_1\delta}{2} - \lambda_1^2(t)\right], \\ \quad t = 0, \dots, T-1, \\ \lambda_1^1(T) = 2(x_1(T) - s_1), \\ \lambda_1^2(T) = 2(x_2(T) - s_1). \end{cases} \tag{5}$$

From the last four equations of system (5) we obtain expressions of $\lambda_1^1(t)$ and $\lambda_1^2(t)$ as functions of x_1 and x_2, $t = 0, \dots, T$. We substitute these expressions of $\lambda_1^1(t)$ and $\lambda_1^2(t)$ into the first equation of system (5) if t belongs to V_1. We get an expression of u_1 as a function of x_1 and x_2. Substituting the new expression of u_1 into equation (1), we get new state equation $x_1(t+1)$ as a function of x_1 and x_2.

Then, we write the Hamiltonian of player 2 as

$$H_2^1\left(x_1(t), x_2(t), \lambda_2^1(t+1), \lambda_2^2(t+1), u_1(t), u_2(t), t\right) =$$
$$= c_2 u_2^2(t) + (x_1(t) - s_2)^2 + (x_2(t) - s_2)^2 +$$
$$+ \delta\lambda_2^1(t+1)\left(x_1(t+1) - x_1(t) - a_1\left(\frac{x_1(t) + x_2(t)}{2} - x_1(t)\right) - u_1(t)\right) +$$
$$+ \delta\lambda_2^2(t+1)\left(x_2(t+1) - x_2(t) - a_2\left(\frac{x_1(t) + x_2(t)}{2} - x_2(t)\right) - u_2(t)\right),$$

for any $t \in V_2$, and

$$\begin{gathered}
H_2^2\left(x_1(t), x_2(t), \lambda_2^1(t+1), \lambda_2^2(t+1), t\right) = \\
= \left(x_1(t) - s_2\right)^2 + \left(x_2(t) - s_2\right)^2 + \\
+ \delta\lambda_2^1(t+1)\left(x_1(t+1) - x_1(t) - a_1\left(\frac{x_1(t) + x_2(t)}{2} - x_1(t)\right)\right) + \\
+ \delta\lambda_2^2(t+1)\left(x_2(t+1) - x_2(t) - a_2\left(\frac{x_1(t) + x_2(t)}{2} - x_2(t)\right)\right),
\end{gathered}$$

for any $t \notin V_2$.

Finding the derivatives $\frac{\partial H_2^1(t)}{\partial u_2(t)} = 0,\ t \in V_2,\ \lambda_2^1(t) = \frac{\partial H_2^1(t)}{\partial x_1(t)} = \frac{\partial H_2^2(t)}{\partial x_1(t)}$ and $\lambda_2^2(t) = \frac{\partial H_2^1(t)}{\partial x_2(t)} = \frac{\partial H_2^2(t)}{\partial x_2(t)},\ t = 1, \ldots, T-1$, we obtain the system

$$\begin{gathered}
\frac{\partial H_2^1(t)}{\partial u_2(t)} = 2c_2 u_2(t) - \delta\lambda_2^2(t+1) = 0,\ t \in V_2, \\
\lambda_2^1(t) = \frac{\partial H_2^1(t)}{\partial x_1(t)} = \frac{\partial H_2^2(t)}{\partial x_1(t)} = \\
= 2\left(x_1(t) - s_2\right) - \delta\lambda_2^1(t+1)\left(1 - \frac{a_1}{2}\right) - \delta\lambda_2^2(t+1)\frac{a_2}{2}, \\
t = 1, \ldots, T-1, \\
\lambda_2^2(t) = \frac{\partial H_2^1(t)}{\partial x_2(t)} = \frac{\partial H_2^2(t)}{\partial x_2(t)} = \\
= 2\left(x_2(t) - s_2\right) - \delta\lambda_2^1(t+1)\frac{a_1}{2} - \delta\lambda_2^2(t+1)\left(1 - \frac{a_2}{2}\right), \\
t = 1, \ldots, T-1, \\
\lambda_2^1(T) = \frac{\partial\left(\left(x_1(T) - s_2\right)^2 + \left(x_2(T) - s_2\right)^2\right)}{\partial x_1(T)} = 2\left(x_1(T) - s_2\right), \\
\lambda_2^2(T) = \frac{\partial\left(\left(x_1(T) - s_2\right)^2 + \left(x_2(T) - s_2\right)^2\right)}{\partial x_2(T)} = 2\left(x_2(T) - s_2\right).
\end{gathered}$$

Finally, we rewrite the system as follows:

$$\begin{cases}
u_2(t) = \frac{\delta}{2c_2}\lambda_2^2(t+1), \quad t \in V_2, \\
\lambda_2^1(t+1) = \frac{2}{\delta(2-a_1)}\left[2\left(x_1(t) - s_2\right) - \lambda_2^1(t) - \lambda_2^2(t+1)\frac{a_2\delta}{2}\right], \\
\quad t = 0, \ldots, T-1, \\
\lambda_2^2(t+1) = \frac{2}{\delta(2-a_2)}\left[2\left(x_2(t) - s_2\right) - \lambda_2^1(t+1)\frac{a_1\delta}{2} - \lambda_2^2(t)\right], \\
\quad t = 0, \ldots, T-1, \\
\lambda_2^1(T) = 2\left(x_1(T) - s_2\right), \\
\lambda_2^2(T) = 2\left(x_2(T) - s_2\right).
\end{cases} \tag{6}$$

We use the same idea as above to find new state equation $x_2(t+1)$ as a function of x_1 and x_2. Taking into account the state equation (2) and (4), we can find the equilibrium state trajectories of agents 1 and 2 according to the initial condition $x_1(0) = x_1^0,\ x_2(0) = x_2^0$. The equilibrium strategy trajectories of players 1 and 2 are also found. Joining two systems (5) and (6) we finish the proof. □

Remark 2. In Theorem, the Nash equilibrium is found under an assumption that the sets of time moments V_1 and V_2, when players 1 and 2 choose their controls, are given. These sets may be different for the players. If we consider the problem of choosing these sets from the optimization problem prespective, then we need to find all possible combinations of time moments for a given number k, and find the Nash equilibrium for any such a pair of sets V_1 and V_2. Moreover, some pair of sets can be preferable (in terms of minimizing the costs) for one player, and another pair can be preferable for another player. Therefore, we could find Pareto optimal sets V_1 and V_2 such that no other pair of sets can give at least the same costs and strictly smaller costs for at least one player. We demonstrate how we find such Pareto-optimal sets V_1 and V_2 for numerical examples in Section 3.

3. Numerical simulations. In this Section, we demonstrate the results of the numerical simulations.

3.1. Numerical example with two moments of control. Let the time horizon be $T = 9$ (periods $0, \ldots, 9$), and $k = 2$ be the number of moments in which players influence agents. The parameters are as follows:

$$a_1 = 0.7,\ a_2 = 0.5,\ \delta = 1,\ c_1 = 0.4,\ c_2 = 0.6,\ s_1 = 0.6,\ s_2 = 0.6,$$

$$x_1(0) = 0.9,\ x_2(0) = 0.1.$$

The initial opinion state of two agents is $x(0) = (0.9,\ 0.1)$, i. e. $x_1(0) = 0.9$, $x_2(0) = 0.1$. The agents 1 and 2 beliefs about the average social opinion are $a_1 = 0.7$, $a_2 = 0.5$, respectively. The discount factor is $\delta = 1$. The unit costs of influence are $c_1 = 0.4$, $c_2 = 0.6$ for players 1 and 2, respestively. Their target opinions are $s_1 = 0.6$, $s_2 = 0.6$. We find the Nash equilibrium for any possible sets V_1 and V_2 consisting of two moments of influence. We obtain that for the sets $V_1 = \{2,\ 5\}$ and $V_2 = \{0,\ 2\}$, both players have the lowest costs in the Nash equilibrium in comparison with all other Nash equilibria. So, this pair of sets V_1 and V_2 is Pareto-optimal. We characterize this equilibrium describing equilibrium strategies and state trajectories (Table 1). The equilibrium costs of players 1 and 2 are 0.3845 and 0.3963, respectively.

Table 1. **Nash equilibrium strategies and state trajectories,**
$V_1 = \{2,\ 5\}$ **and** $V_2 = \{0,\ 2\}$

t	$t_2^1 = 0$	1	$t_1^1 = t_2^2 = 2$	3	4
$x_1(t)$	0.9000	0.6200	0.5577	0.7001	0.6074
$x_2(t)$	0.1000	0.4420	0.3943	0.4352	0.5014
$u_1(t)$			0.1117		
$u_2(t)$	0.1515		0.0745		
t	$t_1^2 = 5$	6	7	8	9
$x_1(t)$	0.5703	0.5410	0.5401	0.5398	0.5396
$x_2(t)$	0.5279	0.5385	0.5391	0.5394	0.5395
$u_1(t)$	-0.0242				
$u_2(t)$					

Agents' opinions are becoming closer to target opinions over time. The equilibrium state and strategy trajectories are shown in Figures 2 and 3.

In this scenario, we observe that among all Nash equilibria when we vary V_1 and V_2, the minimal costs for both players emerge at the Nash equilibrium with $V_1 = \{2,\ 5\}$ and $V_2 = \{0,\ 2\}$.

Remark 3. We calculate the set of all possible time moments V_1 and V_2 by C_T^k combinations for each player. When $T = 9$, and players choose two moments to influence the agents' opinions, then $C_9^2 = 36$. Considering that the two players may have different choices, the number of all possible combinations is $36 \cdot 36 = 1296$. To find the Nash equilibrium, we solved 1296 systems given in Theorem to find the Pareto-optimal equilibrium costs.

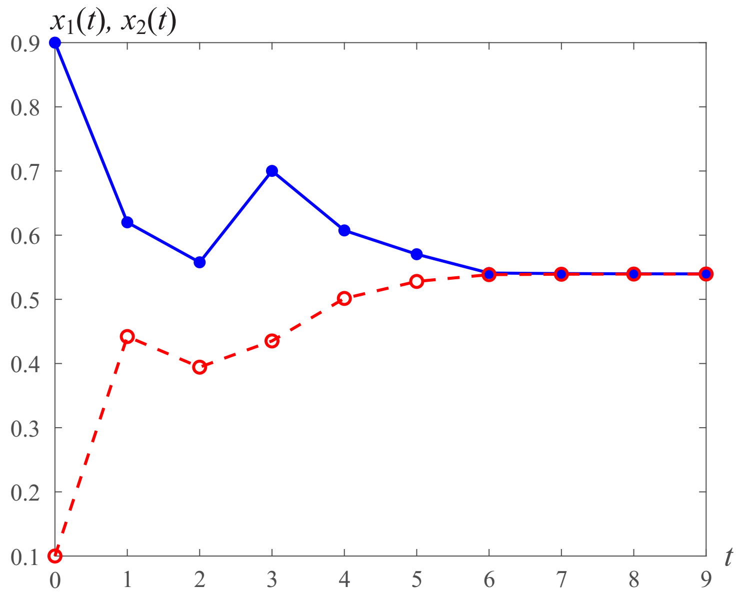

Figure 2. Equilibrium state trajectories (solid — $x_1(t)$, dotted — $x_2(t)$)

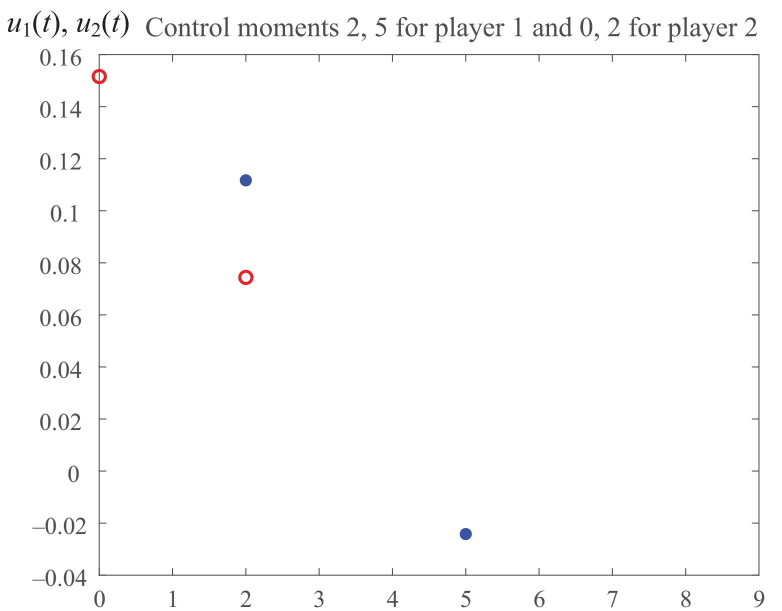

Figure 3. Equilibrium strategy trajectories (solid — $u_1(t)$, dotted — $u_2(t)$)

3.2. Numerical example with three moments of control. Let the time horizon be $T = 8$ (periods $0, \ldots, 8$), and $k = 3$ be the number of moments in which players influence agents. The parameters are as follows:

$$a_1 = 0.1,\ a_2 = 0.7,\ \delta = 1,\ c_1 = 0.4,\ c_2 = 0.6,\ s_1 = 0.2,\ s_2 = 0.1,$$

$$x_1(0) = 0.7,\ x_2(0) = 0.9.$$

The initial opinion state of two agents is $x(0) = (0.7, 0.9)$, i. e. $x_1(0) = 0.7,\ x_2(0) = 0.9$. The agents 1 and 2 beliefs about the average social opinion are $a_1 = 0.1,\ a_2 = 0.7$, respectively. The discount factor is $\delta = 1$. The unit costs of influence are $c_1 = 0.4,\ c_2 = 0.6$ for players 1 and 2, respectively. Their target opinions are $s_1 = 0.2,\ s_2 = 0.1$. We consider all possible sets V_1 and V_2 consisting of three moments when players can influence agents' opinions. Remind that these sets may be different. In this example, we find that different Nash equilibria, i. e. different combinations of sets V_1 and V_2 give the lowest costs to different players. Therefore, the set of Pareto-optimal pairs of sets V_1 and V_2 consists of two elements. We describe both equilibria when (i) $V_1 = \{0,\ 5,\ 7\}$ and $V_2 = \{0,\ 4,\ 5\}$ and (ii) $V_1 = \{0,\ 1,\ 2\}$ and $V_2 = \{0,\ 2,\ 7\}$. Player 1 prefers case (i) and player 2 prefers case (ii). First, we characterize the Nash equilibrium for $V_1 = \{0,\ 5,\ 7\}$ and $V_2 = \{0,\ 4,\ 5\}$ (Table 2). The equilibrium costs of players 1 and 2 are 0.9491 and 1.4566, respectively.

Table 2. **Nash equilibrium strategies and state trajectories, $V_1 = \{0,\ 5,\ 7\}$ and $V_2 = \{0,\ 4,\ 5\}$**

t	$t_1^1 = t_2^1 = 0$	1	2	3	$t_2^2 = 4$
$x_1(t)$	0.7000	0.1584	0.1698	0.1766	0.1807
$x_2(t)$	0.9000	0.3855	0.3060	0.2583	0.2297
$u_1(t)$	-0.5516				
$u_2(t)$	-0.4445				-0.1421
t	$t_1^2 = t_2^3 = 5$	6	$t_1^3 = 7$	8	
$x_1(t)$	0.1831	0.1747	0.1812	0.2100	
$x_2(t)$	0.0704	0.3047	0.2592	0.2319	
$u_1(t)$	-0.0029		0.0249		
$u_2(t)$	0.1948				

Table 3. **Nash equilibrium strategies and state trajectories, $V_1 = \{0,\ 1,\ 2\}$ and $V_2 = \{0,\ 2,\ 7\}$**

t	$t_1^1 = t_2^1 = 0$	$t_1^2 = 1$	$t_1^3 = t_2^2 = 2$	3	4
$x_1(t)$	0.7000	0.0130	0.2241	0.1555	0.1520
$x_2(t)$	0.9000	0.2885	0.1921	0.0842	0.1092
$u_1(t)$	-0.6970	0.1973	-0.0670		
$u_2(t)$	-0.5415		-0.1191		
t	5	6	$t_2^3 = 7$	8	
$x_1(t)$	0.1498	0.1486	0.1478	0.1473	
$x_2(t)$	0.1241	0.1331	0.1385	0.2206	
$u_1(t)$					
$u_2(t)$			0.0789		

Agents' opinions are becoming closer to target opinions over time (Figure 4, *a*). The equilibrium state trajectories and strategy trajectories are shown in Figures 4, *a* and Figure 5, *a*.

Second, we characterize the Nash equilibrium for the case when $V_1 = \{0,\ 1,\ 2\}$ and $V_2 = \{0,\ 2,\ 7\}$, which is preferable for player 2. The equilibrium costs of players 1 and 2 are 1.0462 and 1.2884, respectively. The values of equilibrium state and strategy trajectories are given in Table 3, and they are represented in Figure 4, *b* and Figure 5, *b*.

In order to examine the two Nash equilibria and differences in players' costs, we conducted a comparative analysis in Table 4.

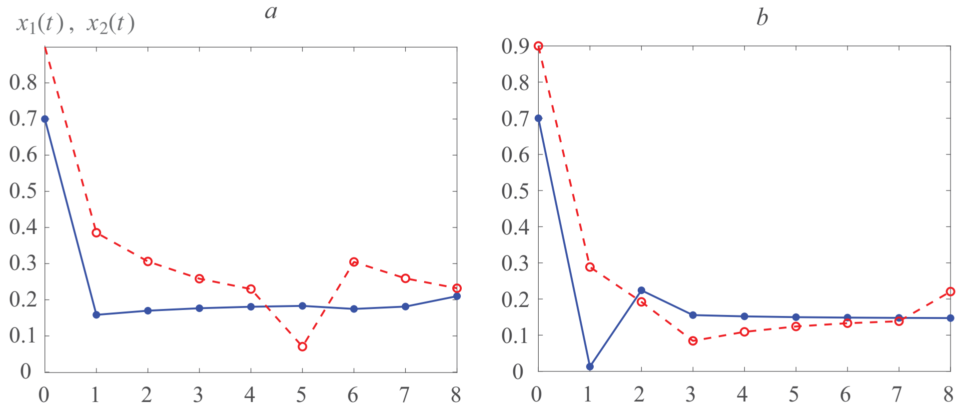

Figure 4. Equilibrium state trajectories, $V_1 = \{0,\ 5,\ 7\}$ and $V_2 = \{0,\ 4,\ 5\}$ (a) and $V_1 = \{0,\ 1,\ 2\}$ and $V_2 = \{0,\ 2,\ 7\}$ (b) (solid — $x_1(t)$, dotted — $x_2(t)$)

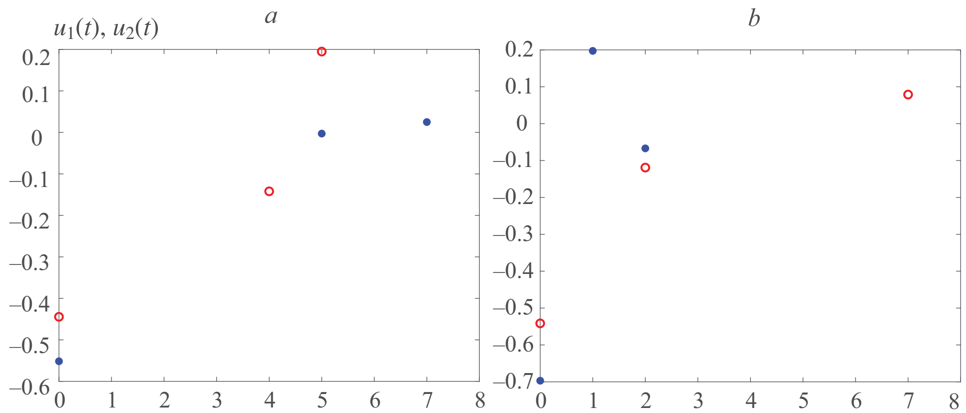

Figure 5. Equilibrium strategies, $V_1 = \{0,\ 5,\ 7\}$ and $V_2 = \{0,\ 4,\ 5\}$ (a) and $V_1 = \{0,\ 1,\ 2\}$ and $V_2 = \{0,\ 2,\ 7\}$ (b) (solid — $u_1(t)$, dotted — $u_2(t)$)

Table 4. **Comparison of the two Nash equilibria**

Player	Time set, $equil_1$	Costs	Time set, $equil_2$	Costs	Index, %
1	$\{0, 5, 7\}$	0.9491*	$\{0, 1, 2\}$	1.0462	10.23
2	$\{0, 4, 5\}$	1.4566	$\{0, 2, 7\}$	1.2884*	13.05

In Table 4, the costs with an asterisk are minimal for the corresponding player. To estimate the difference between two Nash equilibria we use the following index for player 1:

$$\frac{J_1(equil_2) - J_1^*(equil_1)}{J_1^*\,(equil_1)} \cdot 100 = 10.23\ \%,$$

where by $equil_1$ and $equil_2$ we mean the Nash equilibria with time sets $V_1 = \{0,\ 5,\ 7\}$, $V_2 = \{0,\ 4,\ 5\}$ and $V_1 = \{0,\ 1,\ 2\}$, $V_2 = \{0,\ 2,\ 7\}$, respectively. Although this value is not very large, it implies that in the second Nash equilibrium, the cost loss borne by player 1 is not too high compared to the first Nash equilibrium. Similarly, for player 2 the index comparing two Nash equilibria is

$$\frac{J_2(equil_1) - J_2^*(equil_2)}{J_2^*(equil_2)} \cdot 100 = 13.05\ \%.$$

We could notice that player 2 does not bear much costs in the first Nash equilibrium relative to the second one.

To sum up, Theorem provides the necessary conditions for the Nash equilibrium for a competition game when the moments of controls are fixed. As we showed above, some equilibrium may be preferable for one player while not for another player. In this case, if players have an option to choose sets V_1 and V_2, when they control the agents, there may be a conflict of interests between the players. We do not discuss how one of the Nash equilibria can be chosen, but it can be modeled as a bargaining process.

4. Conclusions. In this paper a model of opinion dynamics where agents' opinions are influenced by the players is proposed. The players are willing to minimize their costs which are represented by the sum of squared distances of the agents' opinions from the desired opinion and quadratic functions of controls. The main feature of the model is that the players can influence agents' opinions in a limited number of time moments. We find the Nash equilibrium in the game in which the number of such moments is given and it is the same for both players. We also find Pareto-optimal sets of time moments in our numerical simulations.

References

1. Degroot M. H. Reaching a consensus. *Journal of the American Statistical Assocition*, 1974, vol. 69, iss. 345, pp. 118–121.
2. Friedkin N. E., Johnsen E. C. Social influence and opinions. *Journal of Mathematical Sociology*, 1990, vol. 15, pp. 193–206.
3. Deffuant G., Neau D., Amblard F., Weisbuch G. Mixing beliefs among interacting agents. *Advances in Complex Systems*, 2000, vol. 3, no. 1–4, pp. 87–98.
4. Gao J., Parilina E. M. Average-oriented opinion dynamics with the last moment of observation. *Control Processes and Stability*, 2021, vol. 8, no. 1, pp. 505–509.
5. Gao J., Parilina E. M. Opinion control problem with average-oriented opinion dynamics and limited observation moments. *Contributions to Game Theory and Management*, 2021, vol. 14, pp. 103–112.
6. Gao J., Parilina E. M. Opinion dynamics in multiagent systems with optimal choice of opinion verification moments. *Doklady Mathematics*, 2023, vol. 108, suppl. 1, pp. S75–S85. https://doi.org/10.1134/S1064562423600732
7. Gao J., Parilina E. M. Optimal control in a multiagent opinion dynamic system. *Contributions to Game Theory and Management*, 2022, vol. 15, pp. 51–59.
8. Lang N., Zha Q., Wang L. Competitive targeted marketing in social networks with switching topology: seed selection and consensus shaping. *Information Fusion*, 2023, vol. 95, pp. 355–371.
9. Goyal S., Kearns M. Competitive contagion in networks. *Proceedings of the Forty-Fourth Annual ACM Symposium on Theory of Computing*, 2012, pp. 759–774.
10. Carnes T., Nagarajan C., Wild S. M., van Zuylen A. Maximizing influence in a competitive social network: a follower' s perspective. *Proceedings of the Ninth International Conference on Electronic Commerce*, 2007, pp. 351–360.
11. Mazalov V. V., Parilina E. M. Game of competition for opinion with two centers of influence. *Mathematical Optimization Theory and Operations Research*, 2019, pp. 673–684.
12. Goyal M., Manjunath D. Opinion control competition in a social network. *2020 International Conference on COMmunication Systems & NETworkS (COMSNETS)*, 2020, pp. 306–313.
13. Liu Y., Liu J., Wu K. Cost-effective competition on social networks: a multi-objective optimization perspective. *Information Sciences*, 2023, vol. 620, pp. 31–46.
14. He G., Ruan H., Wu Y., Liu J. Opinion dynamics with competitive relationship and switching topologies. *IEEE Access*, 2021, vol. 9, pp. 3016–3025.
15. Mandel A., Venel X. Dynamic competition over social networks. *European Journal of Operational Research*, 2020, vol. 280, no. 2, pp. 597–608.

16. Yu J., Wang Y., Li J., Shen H., Cheng X. Analysis of competitive information dissemination in social network based on evolutionary game model. *2012 Second International Conference on Cloud and Green Computing*, 2012, pp. 748–753.

17. Goyal M., Chatterjee D., Karamchandani N., Manjunath D. Maintaining ferment. *2019 IEEE* 58^{th} *Conference on Decision and Control (CDC)*, 2019, pp. 5217–5222.

18. Gao J. Two-player opinion control game with limited observation moments. *Control Processes and Stability*, 2022, vol. 9, no. 1, pp. 464–468.

19. Sedakov A. A., Zhen M. Opinion dynamics game in a social network with two influence nodes. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2019, vol. 15, iss. 1, pp. 118–125. https://doi.org/10.21638/11701/spbu10.2019.109

20. Kareeva Y., Sedakov A. A., Zhen M. Influence in social networks with stubborn agents: from competition to bargaining. *Applied Mathematics and Computation*, 2023, vol. 444, pp. 127790. https://doi.org/10.1016/j.amc.2022.127790

21. Jiang H., Mazalov V. V., Gao H., Wang C. Opinion dynamics control in a social network with a communication structure. *Dynamic Games and Applications*, 2023, vol. 13, no. 1, pp. 412–434. https://doi.org/10.1007/s13235-021-00406-y

22. Haurie A., Krawczyk J. B., Zaccour G. *Games and dynamic games.* Singapore, World Scientific Publishing Company, 2012, 488 p.

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Конкуренция за мнения агентов в небольших динамических системах с ограниченным управлением*

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Исследуется динамическая система конкурентного мнения, представляющая модель социальной сети, развивающейся во времени. Агенты сравнивают свои мнения со средними и формируют новые мнения. Существуют игроки, которые контролируют мнение агентов в социальной сети, и их цель — приблизить мнения в сети к желаемому уровню в течение конечного промежутка времени, минимизируя свои затраты. Особенность представленной модели в том, что игроки могут влиять на агентов в ограниченное число моментов времени. На числовых примерах показано, как игроки могут выбирать моменты влияния из оптимального по Парето множества. Приведены результаты численного моделирования.

Ключевые слова: конкуренция за мнения, равновесие по Нэшу, динамика мнений, социальная сеть.

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