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# On the boundary control problem for a pseudo-parabolic equation with involution

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Previously, some control problems for the pseudo-parabolic equation independent of involution were considered. In this paper, we consider a boundary control problem associated with a pseudo-parabolic equation with involution in a bounded one-dimensional domain. On the part of the border of the considered domain, the value of the solution with control function is given. Restrictions on the control are given in such a way that the average value of the solution in the considered domain gets a given value. The problem given by the method of separation of variables is reduced to the Volterra integral equation of the second kind. The existence of the control function was proved by the Laplace transform method.

*Keywords*: boundary problem, Volterra integral equation, control function, Laplace transform, involution.

**1. Introduction.** In this article, we consider the following pseudo-parabolic equation with involution in the domain $\Omega_T := (0,\pi)\times(0,\infty)$:

$$u_t(x,t) - u_{xx}(x,t) + \varepsilon\, u_{xx}(\pi - x,t) - u_{xxt}(x,t) + \varepsilon\, u_{xxt}(\pi - x,t) \;=\; 0, \quad (x,t)\in\Omega_T, \tag{1}$$

with boundary value conditions

$$u(0,t) \;=\; \nu(t), \quad u(\pi,t) \;=\; 0, \quad t \geqslant 0, \tag{2}$$

and initial condition

$$u(x,0) \;=\; 0, \quad 0 \leqslant x \leqslant \pi, \tag{3}$$

here $\varepsilon$ is a nonzero real number such that $|\varepsilon| < 1$ and $\nu(t)$ is the control function, which gives the flow amplitude.

**Definition 1.** *If the control function $\nu(t) \in W_2^1(\mathbb{R}_+)$ satisfies the conditions $\nu(0) = 0$ and $|\nu(t)| \leqslant 1$ on the half-line $t \geqslant 0$, we call it admissible control.*

We will prove later in Section 3 that the function $\nu$ belongs to the class $W_2^1(\mathbb{R}_+)$.

Differential equations with modified arguments are equations in which the un-known function and its derivatives are evaluated with modifications of time or space variables; such equations are called, in general, functional differential equations. Among such equations, one can single out, equations with involutions [1].

**Definition 2** ([2, 3]). *A function $f(x) \not\equiv x$ maps bijectively a set of real numbers $\Omega$, such that*

$$f(f(x)) \;=\; x \text{ or } f^{-1}(x) \;=\; f(x),$$

*is called an involution on $\Omega$.*

Due to the widespread use of partial differential equations in physics and technology, there is always a great interest in the study of boundary value control problems. For this purpose, various boundary problems for parabolic and pseudo-parabolic equations have been widely studied by many researchers.

It can be seen that equation (1) for $\varepsilon = 0$ is a classical pseudo-parabolic equation. If $\varepsilon \neq 0$, equation (1) relates the values of the second derivatives at two different points and becomes a nonlocal equation. It is known that boundary control problems for the pseudo-parabolic equation in the case $\varepsilon = 0$ are studied in detail in works [4, 5].

The pseudo-parabolic type equations arise in areas such as fluid flow [6], heat transfer [7], and the diffusion of radiation [8]. Roughly speaking, pseudo-parabolic equations account for higher order correction in the model than do parabolic equations. The boundary control of a pseudo-parabolic equation and compare the results to those of parabolic equations was studied in [9]. The stability, uniqueness, and existence of solutions of some classical problems for the considered equation are studied in [10]. In [11], the point control problems for pseudo-parabolic and parabolic type equations are considered. In [12], some problems related to distributed parameter impulse control problems for systems were studied.

We now consider the following control problem.

**Control problem.** *Suppose that the function $\varphi(t)$ is given. Then find the control function $\nu(t)$ from the equation*

$$\int_0^\pi u(x,t)\,dx \;=\; \varphi(t), \quad t \geqslant 0, \tag{4}$$

*here $u(x,t)$ is the solution of the mixed problem* (1)–(3).

The physical meaning of the equation (4) is the average temperature in the rod.

In [13, 14], the optimal control problem for the parabolic type equations was studied. Control problems for the infinite-dimensional case were studied by Egorov [15], who generalized Pontryagin's maximum principle to a class of equations in Banach space, and the proof of a bang-bang principle was shown in the particular conditions.

The boundary control problem for a parabolic equation with a piecewise smooth boundary in an $n$-dimensional domain was studied in [16] and an estimate for the minimum time required to reach a given average temperature was found. Control problems for the heat transfer equation in the three-dimensional domain are studied in [17].

Control problems for parabolic equations in bounded one and two-dimensional domains are studied in works [18–21]. In these articles, an estimate was found for the minimum time required to heat a bounded domain to an estimate average temperature. The existence of control function is proved by Laplace transform method.

A lot of information on the optimal control problems was given in detail in the monographs of Lions and Fursikov [22, 23]. Practical approaches to general numerical optimization and optimal control for equations of the second order parabolic type are studied in works such as [24, 25].

It is known that in recent years, due to the increasing interest in physics and mathematics, the boundary problems related to heat diffusion equations related to involution are widely studied. Note that in recent years, there have been many papers on spectral problems for involved differential operators (see [26]). In [27], the classical solution of the initial-boundary value problem for the first-order partial differential equation with involution in the function and its derivative was determined by the Fourier method.

In [28], a boundary value problem for the heat equation associated with involution in a one-dimensional domain is studied. Many boundary value problems for parabolic and pseudo-parabolic equations were studied in works [29–31].

In this work, the boundary control problem for the pseudo-parabolic equation with involution is considered. The boundary control problem studied in this work is reduced to the Volterra integral equation of the second kind by the Fourier method (Section 2). In Section 3, the existence of a solution to the integral equation is proved using the Laplace transform method.

**2. Volterra integral equation.** In this Section, we consider how the given control problem can be reduced to a Volterra integral equation of the second kind.

We now consider the spectral problem

$$X''(x) - \varepsilon X''(\pi - x) + \lambda X(x) \; = \; 0, \quad 0 < x < \pi,$$

$$X(0) = X(\pi) = 0, \quad 0 \leqslant x \leqslant \pi,$$

here $|\varepsilon| < 1$, $\varepsilon \in \mathbb{R} \setminus \{0\}$. It is proved in [29, 30] that expressing the solution of spectral problem in terms of the sum of even and odd functions, one finds the following eigenvalues:

$$\lambda_{2k} = 4\,(1+\varepsilon)\,k^2, \quad k \in \mathbb{N}, \tag{5}$$

$$\lambda_{2k+1} = (1-\varepsilon)\,(2k+1)^2, \quad k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \tag{6}$$

and we have the eigenfunctions

$$X_{2k} = \sin 2kx, \quad k \in \mathbb{N}, \quad X_{2k+1} = \sin(2k+1)x, \quad k \in \mathbb{N}_0.$$

By the solution of the problem (1)–(3) we understand the function $u(x,t)$, which is given in the form

$$u(x,t) = \nu(t)\,\frac{\pi - x}{\pi} - w(x,t), \tag{7}$$

where the function $w(x,t) \in C^{2,1}_{x,t}(\Omega_T) \cap C(\bar{\Omega}_T)$ is the solution to the mixed problem

$$w_t(x,t) - w_{xx}(x,t) + \varepsilon\, w_{xx}(\pi - x, t) - w_{xxt}(x,t) + \varepsilon\, w_{xxt}(\pi - x, t) \; = \; \frac{\pi - x}{\pi}\,\nu'(t),$$

with initial-boundary conditions

$$w(0,t) = w(\pi,t) = 0, \quad w(x,0) = 0.$$

We solve the solution of the above mixed problem by the Fourier method.

Thus, we obtain (see [32])

$$w(x,t) \; = \; \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{1}{2k+1}\,\frac{1}{1+\lambda_{2k+1}} \left( \int\limits_0^t e^{-\mu_{2k+1}(t-s)}\,\nu'(s)\,ds \right) \sin(2k+1)x \, +$$

$$+ \; \frac{2}{\pi}\sum_{k=1}^{\infty} \frac{1}{2k}\,\frac{1}{1+\lambda_{2k}} \left( \int\limits_0^t e^{-\mu_{2k}(t-s)}\,\nu'(s)\,ds \right) \sin 2kx, \tag{8}$$

here $\mu_k = \frac{\lambda_k}{1+\lambda_k}$.

It follows from (7) and (8), we get the solution of the mixed problem (1)–(3):

$$u(x,t) = \frac{\pi - x}{\pi}\,\nu(t) \; -$$

$$- \; \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\,\frac{1}{1+\lambda_{2k+1}}\left(\int\limits_0^t e^{-\mu_{2k+1}(t-s)}\,\nu'(s)\,ds\right)\sin(2k+1)x \; -$$

$$- \; \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k}\,\frac{1}{1+\lambda_{2k}}\left(\int\limits_0^t e^{-\mu_{2k}(t-s)}\,\nu'(s)\,ds\right)\sin 2kx. \qquad (9)$$

Using the condition (4) and the solution (9), we can write

$$\varphi(t) = \int\limits_0^{\pi} u(x,t)\,dx = \nu(t)\int\limits_0^{\pi}\frac{\pi - x}{\pi}\,dx \; -$$

$$- \; \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}\,\frac{1}{1+\lambda_{2k+1}}\int\limits_0^t e^{-\mu_{2k+1}(t-s)}\,\nu'(s)\,ds\int\limits_0^{\pi}\sin(2k+1)x\,dx \; -$$

$$- \; \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1}{2k}\,\frac{1}{1+\lambda_{2k}}\int\limits_0^t e^{-\mu_{2k}(t-s)}\,\nu'(s)\,ds\int\limits_0^{\pi}\sin 2kx\,dx.$$

Then we may write

$$\varphi(t) = \; \nu(t)\int\limits_0^{\pi}\frac{\pi - x}{\pi}\,dx \; -$$

$$- \; \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{1-(-1)^{2k+1}}{(2k+1)^2}\,\frac{1}{1+\lambda_{2k+1}}\int\limits_0^t e^{-\mu_{2k+1}(t-s)}\,\nu'(s)\,ds \; -$$

$$- \; \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{1-(-1)^{2k}}{4k^2}\,\frac{1}{1+\lambda_{2k}}\int\limits_0^t e^{-\mu_{2k}(t-s)}\,\nu'(s)\,ds.$$

It is clear that $1-(-1)^{2k+1} = 2$ and $1-(-1)^{2k} = 0$ for $k = 0, 1, \ldots$. Then we get

$$\varphi(t) = \; \nu(t)\int\limits_0^{\pi}\frac{\pi - x}{\pi}\,dx \; -$$

$$- \; \frac{2}{\pi}\sum_{k=0}^{\infty}\frac{2}{(2k+1)^2}\,\frac{1}{1+\lambda_{2k+1}}\int\limits_0^t e^{-\mu_{2k+1}(t-s)}\,\nu'(s)\,ds. \qquad (10)$$

According to the definition of the control function $\nu(t)$ and (10), we can write

$$\varphi(t) \; = \; \nu(t)\int\limits_0^{\pi}\frac{\pi - x}{\pi}\,dx - \nu(t)\,\frac{2}{\pi}\sum_{k=0}^{\infty}\frac{2}{(2k+1)^2}\,\frac{1}{1+\lambda_{2k+1}} \; +$$

$$+ \frac{4(1-\varepsilon)}{\pi} \sum_{k=0}^{\infty} \frac{1}{(1+\lambda_{2k+1})^2} \int_0^t e^{-\mu_{2k+1}(t-s)} \nu(s)\, ds.$$

According to Parseval equation,

$$\int_0^\pi \frac{\pi - x}{\pi}\, dx = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}.$$

As a result we have

$$\varphi(t) = \nu(t)\, \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\mu_{2k+1}}{(2k+1)^2} +$$

$$+ \frac{4(1-\varepsilon)}{\pi} \sum_{k=0}^{\infty} \frac{1}{(1+\lambda_{2k+1})^2} \int_0^t e^{-\mu_{2k+1}(t-s)} \nu(s)\, ds,$$

here $0 < \mu_{2k+1} = \frac{\lambda_{2k+1}}{1+\lambda_{2k+1}} < 1$.

We set

$$K(t) = \sum_{k=0}^{\infty} \beta_{2k+1}\, e^{-\mu_{2k+1}\, t}, \quad t > 0, \tag{11}$$

and

$$\alpha = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{\mu_{2k+1}}{(2k+1)^2}, \tag{12}$$

where $\beta_{2k+1}$ is defined as

$$\beta_{2k+1} = \frac{4(1-\varepsilon)}{\pi} \frac{1}{(1+\lambda_{2k+1})^2}, \quad k \in \mathbb{N}_0. \tag{13}$$

Thus, we have the following Volterra integral equation of the second kind:

$$\alpha\, \nu(t) + \int_0^t K(t-s)\, \nu(s) ds = \varphi(t), \quad t > 0. \tag{14}$$

Assume that $M > 0$ is constant. Then we denote by $W(M)$ the set of function $\varphi \in W_2^2(-\infty, +\infty)$, $\varphi(t) = 0$ for all $t \leqslant 0$ which satisfying the condition

$$\|\varphi\|_{W_2^2(R_+)} \leqslant M.$$

We present the following main theorem.

**Theorem.** *There exists $M > 0$ such that for any function $\varphi \in W(M)$ the solution $\nu(t)$ of the Volterra integral equation* (14) *exists and satisfies condition $|\nu(t)| \leqslant 1$.*

**Proposition 1.** *Assume that $\varepsilon$ is a nonzero real number such that $|\varepsilon| < 1$. Then, the kernel $K(t)$ of the integral equation* (14) *is continuous on the half-line $t \geqslant 0$.*

P r o o f. According to (5) and (6)

$$\lambda_{2k+1} = (1-\varepsilon)\,(2k+1)^2, \quad k \in \mathbb{N}_0.$$

Then from (13)

$$0 < \beta_{2k+1} = \frac{4\,(1-\varepsilon)}{\pi}\,\frac{1}{(1+\lambda_{2k+1})^2}.$$

Thus, from (11) we have

$$0 < K(t) \;=\; \sum_{k=0}^{\infty} \beta_{2k+1}\, e^{-\mu_{2k+1}\,t} \leqslant \frac{4\,(1-\varepsilon)}{\pi} \sum_{k=0}^{\infty} \frac{1}{(1+\lambda_{2k+1})^2} = C_\varepsilon,$$

here $C_\varepsilon$ is a constant only depending on $\varepsilon$. □

**3. Existence of control function.** In this Section, we will consider at the existence of the control function.

Now, we rewrite the integral equation (14)

$$\alpha\,\nu(t) + \int_0^t K(t-s)\,\nu(s)ds = \varphi(t), \quad t > 0.$$

We find the solution of this Volterra integral equation using the method of Laplace transform. We know that

$$\widetilde{\nu}(p) = \int_0^\infty e^{-pt}\,\nu(t)\,dt.$$

Then we use Laplace transform obtain the equation

$$\widetilde{\varphi}(p) = \alpha \int_0^\infty e^{-pt}\,\nu(t)dt + \int_0^\infty e^{-pt}\,dt \int_0^t K(t-s)\,\nu(s)ds \;=$$

$$=\; \alpha\,\widetilde{\nu}(p) + \widetilde{K}(p)\,\widetilde{\nu}(p).$$

Thus, we obtain

$$\widetilde{\nu}(p) = \frac{\widetilde{\varphi}(p)}{\alpha + \widetilde{K}(p)},$$

where $p = \xi + i\,\tau,\ \xi > 0,\ \tau \in \mathbb{R}$, and we can write the function $\nu(t)$ as

$$\nu(t) = \frac{1}{2\pi i}\int_{\xi-i\,\infty}^{\xi+i\,\infty} \frac{\widetilde{\varphi}(p)}{\alpha+\widetilde{K}(p)}\,e^{pt}dp = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\widetilde{\varphi}(\xi+i\,\tau)}{\alpha+\widetilde{K}(\xi+i\,\tau)}\,e^{(\xi+i\,\tau)t}\,d\tau. \tag{15}$$

**Proposition 2.** *The following estimate is valid:*

$$|\alpha + \widetilde{K}(\xi+i\,\tau)| \geqslant \frac{C_\xi}{\sqrt{1+\tau^2}}, \quad \xi > 0, \quad \tau \in \mathbb{R},$$

*here* $\alpha > 0$ *is defined by* (12) *and* $C_\xi > 0$ *is a constant only depending on* $\xi$.

P r o o f. It is clear that $\alpha > 0$, which defined by (12). Using the Laplace transform, we may write

$$\widetilde{K}(p) = \int_0^\infty K(t)\,e^{-pt}\,dt = \sum_{k=0}^{\infty} \beta_{2k+1} \int_0^\infty e^{-(p+\mu_{2k+1})t}\,dt \;=$$

$$= \sum_{k=0}^{\infty} \frac{\beta_{2k+1}}{p+\mu_{2k+1}},$$

where $K(t)$ defined by (11) and

$$\alpha + \widetilde{K}(\xi + i\,\tau) \;=\; \sum_{k=0}^{\infty} \frac{\beta_{2k+1}}{\xi+\mu_{2k+1}+i\,\tau} \;=$$

$$=\; \alpha + \sum_{k=0}^{\infty} \frac{\beta_{2k+1}\,(\xi+\mu_{2k+1})}{(\xi+\mu_{2k+1})^2+\tau^2} - i\,\tau \sum_{k=0}^{\infty} \frac{\beta_{2k+1}}{(\xi+\mu_{2k+1})^2+\tau^2} \;=$$

$$=\; \mathrm{Re}(\alpha + \widetilde{K}(\xi + i\,\tau)) + i\,\mathrm{Im}(\alpha + \widetilde{K}(\xi + i\,\tau)),$$

$\mathrm{Re}(\alpha + \widetilde{K}(\xi + i\,\tau))$ and $\mathrm{Im}(\alpha + \widetilde{K}(\xi + i\,\tau))$ are defined as

$$\mathrm{Re}(\alpha + \widetilde{K}(\xi + i\,\tau)) = \alpha + \sum_{k=0}^{\infty} \frac{\beta_{2k+1}\,(\xi+\mu_{2k+1})}{(\xi+\mu_{2k+1})^2+\tau^2},$$

$$\mathrm{Im}(\alpha + \widetilde{K}(\xi + i\,\tau)) = -\tau \sum_{k=0}^{\infty} \frac{\beta_{2k+1}}{(\xi+\mu_{2k+1})^2+\tau^2}.$$

We know that

$$(\xi+\mu_{2k+1})^2+\tau^2 \leqslant [(\xi+\mu_{2k+1})^2+1](1+\tau^2),$$

and we get

$$\frac{1}{(\xi+\mu_{2k+1})^2+\tau^2} \geqslant \frac{1}{1+\tau^2}\,\frac{1}{(\xi+\mu_{2k+1})^2+1}. \tag{16}$$

Thus, according to (16) we can obtain the following estimates:

$$|\mathrm{Re}(\alpha + \widetilde{K}(\xi + i\,\tau))| = \alpha + \sum_{k=0}^{\infty} \frac{\beta_{2k+1}\,(\xi+\mu_{2k+1})}{(\xi+\mu_{2k+1})^2+\tau^2} \geqslant$$

$$\geqslant \frac{1}{1+\tau^2} \sum_{k=0}^{\infty} \frac{\beta_{2k+1}\,(\xi+\mu_{2k+1})}{(\xi+\mu_{2k+1})^2+1} = \frac{C_{1,\xi}}{1+\tau^2} \tag{17}$$

and

$$|\mathrm{Im}(\alpha + \widetilde{K}(\xi + i\,\tau))| = |\tau| \sum_{k=0}^{\infty} \frac{\beta_{2k+1}}{(\xi+\mu_{2k+1})^2+\tau^2} \geqslant$$

$$\geqslant \frac{|\tau|}{1+\tau^2} \sum_{k=0}^{\infty} \frac{\beta_{2k+1}}{(\xi+\mu_{2k+1})^2+1} = \frac{C_{2,\xi}\,|\tau|}{1+\tau^2}. \tag{18}$$

Parameters $C_{1,\xi}$, $C_{2,\xi}$ are constants only depending on $\xi$ and are as

$$C_{1,\xi} = \sum_{k=0}^{\infty} \frac{\beta_{2k+1}\,(\xi+\mu_{2k+1})}{(\xi+\mu_{2k+1})^2+1}, \quad C_{2,\xi} = \sum_{k=0}^{\infty} \frac{\beta_{2k+1}}{(\xi+\mu_{2k+1})^2+1}.$$

From (17) and (18), we have the estimates

$$|\alpha + \widetilde{K}(\xi + i\,\tau)|^2 = |\mathrm{Re}(\alpha + \widetilde{K}(\xi + i\,\tau))|^2 + |\mathrm{Im}(\alpha + \widetilde{K}(\xi + i\,\tau))|^2 \geqslant$$

$$\geqslant \frac{\min(C_{1,\xi}^2, C_{2,\xi}^2)}{1+\tau^2}$$

and

$$|\alpha + \widetilde{K}(\xi + i\,\tau)| \geqslant \frac{C_\xi}{\sqrt{1+\tau^2}}, \tag{19}$$

here $C_\xi = \min(C_{1,\xi}, C_{2,\xi})$. □

Then, proceeding to the limit as $\xi \to 0$ from (15), we obtain the equality

$$\nu(t) = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{\widetilde{\varphi}(i\,\tau)}{\alpha + \widetilde{K}(i\,\tau)}\, e^{i\,\tau t}\, d\tau. \tag{20}$$

**Proposition 3** [19]. *Assume that $\varphi(t) \in W(M)$. Then for the imaginary part of the Laplace transform of function $\varphi(t)$ the inequality is hold*

$$\int\limits_{-\infty}^{+\infty} |\widetilde{\varphi}(i\,\tau)| \sqrt{1+\tau^2}\, d\tau \leqslant C\, \|\varphi\|_{W_2^2(R_+)},$$

*where $C > 0$ is a constant.*

Now we prove Theorem.

P r o o f of Theorem. First of all, we prove that $\nu \in W_2^1(\mathbb{R}_+)$. Due to (19) and (20), we get

$$\int\limits_{-\infty}^{+\infty} |\widetilde{\nu}(\tau)|^2 (1+|\tau|^2)\, d\tau \;=\; \int\limits_{-\infty}^{+\infty} \left| \frac{\widetilde{\varphi}(i\,\tau)}{\alpha + \widetilde{K}(i\,\tau)} \right|^2 (1+|\tau|^2)\, d\tau \;\leqslant$$

$$\leqslant\; C_0 \int\limits_{-\infty}^{+\infty} |\widetilde{\varphi}(i\,\tau)|^2 (1+|\tau|^2)^2\, d\tau \;=\; C_0 \|\varphi\|_{W_2^2(\mathbb{R})}^2,$$

here $C_0 = \min(C_{1,0}, C_{2,0})$ which is defined by (19).

Besides, we have

$$|\nu(t) - \nu(s)| \;=\; \left| \int\limits_s^t \nu'(y)\, dy \right| \;\leqslant\; \|\nu'\|_{L_2} (t-s)^{1/2}.$$

From (19), (20) and Proposition 3, we can write

$$|\nu(t)| \leqslant \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{|\widetilde{\varphi}(i\,\tau)|}{|\alpha + \widetilde{K}(i\,\tau)|} d\tau \leqslant$$

$$\leqslant\; \frac{1}{2\pi C_0} \int\limits_{-\infty}^{+\infty} |\widetilde{\varphi}(i\,\tau)| \sqrt{1+\tau^2} d\tau \;\leqslant$$

$$\leqslant\; \frac{C}{2\pi C_0} \|\varphi\|_{W_2^2(R_+)} \;\leqslant \frac{C\,M}{2\pi C_0} = 1,$$

where $M = \frac{2\pi C_0}{C}$. □

**4. Example.** We consider the following function:

$$\varphi(t) = \begin{cases} 0, & \text{for} \quad t \leqslant 0; \\ H\,t^2\,e^{-t}, & \text{for} \quad t > 0, \end{cases} \tag{21}$$

where $H > 0$ is a constant number. The physical meaning of the function $\varphi(t)$ is the average temperature in the rod.

Suppose $\varepsilon = \frac{1}{2}$ in equation (1). Then the function $K(t)$ and $\alpha$ determined by (11) and (12) are as follows:

$$K(t) = \frac{8}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2 + (2k+1)^2)^2}\, e^{-\mu_{2k+1}\, t}, \quad t > 0,$$

here $\mu_{2k+1} = \frac{(2k+1)^2}{2+(2k+1)^2}$, and

$$\alpha = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2 + (2k+1)^2}. \tag{22}$$

We can represent the kernel $K(t)$ in the form

$$K(t) = \frac{8}{9\,\pi}\, e^{-\mu_1\, t} + \frac{8}{\pi} \sum_{k=1}^{\infty} \frac{1}{(2 + (2k+1)^2)^2}\, e^{-\mu_{2k+1}\, t} =$$

$$= e^{-\mu_1\, t} \Big( \frac{8}{9\,\pi} + O(1)\, e^{-t(\mu_3 - \mu_1)} \Big), \quad \mu_1 = \frac{1}{3}.$$

Consequently, we can write $K(t) \simeq \frac{8}{9\,\pi}\, e^{-\frac{1}{3}\, t}$ for $t > 0$.

In this case, the main integral equation (14) can be replaced by the approximation

$$\alpha\, \nu(t) + \frac{8}{9\,\pi} \int_0^t e^{-\frac{1}{3}\,(t-s)}\, \nu(s) ds = H\, t^2\, e^{-t}, \quad t > 0.$$

We obtain the following solution using the Laplace transform

$$\widetilde{\nu}(p) = \frac{2\,H}{\alpha}\, \frac{3p+1}{(3p+A)\,(p+1)^3},$$

here $A = 1 + \frac{8}{3\pi}\, \frac{1}{\alpha} > 0$.

From (22) we can write

$$\alpha = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{1}{2 + (2k+1)^2} > \frac{4}{3\,\pi}.$$

Then,

$$A = 1 + \frac{8}{3\pi}\, \frac{1}{\alpha} < 1 + \frac{8}{3\pi}\, \frac{3\pi}{4} = 3.$$

Thus, we define the original function $\nu(t)$, which is the solution of the integral equation, as

$$\nu(t) = \frac{2\,H}{\alpha}\, \frac{9\,(1-A)}{(3-A)^3} e^{-\frac{A}{3}\, t} -$$

$$- \frac{2\,H\,e^{-t}}{\alpha}\left(\frac{1}{A-3}t^2+\frac{3\,(1-A)}{(A-3)^2}t-\frac{9\,(1-A)}{(A-3)^3}\right). \tag{23}$$

It is known that $\nu(0)=0$. Set

$$g(t)=\frac{2}{\alpha}\,\frac{9\,(1-A)}{(3-A)^3}e^{-\frac{A}{3}t}-$$

$$-\frac{2\,e^{-t}}{\alpha}\left(\frac{1}{A-3}t^2+\frac{3\,(1-A)}{(A-3)^2}t-\frac{9\,(1-A)}{(A-3)^3}\right), \quad t>0.$$

Note that $\lim_{t\to\infty} g(t)=0$. Let the function $|g(t)|$ reach its maximum value at the point $T^*$. Therefore, we can write

$$\max|g(t)|=|g(T^*)|=B,$$

here $B=\text{const}>0$.

If we take as $H\leqslant\frac{1}{B}$, then we have the following estimate:

$$|\nu(t)|\leqslant H\,|g(t)|\leqslant HB\leqslant 1.$$

Thus, when the average temperature in the rod is given by equation (21), we found the control function $\nu(t)$ in the form (23) and verified that it is admissible.

**5. Conclusion.** In this paper, a boundary value control problem for a pseudo-parabolic equation with involution is considered. The studied boundary control problem is reduced to the Volterra integral equation of the second kind by the Fourier method, and the existence of a solution of the integral equation is proved using the Laplace transform method. In future research, we will continue to improve and study boundary-value control problems for pseudo-parabolic equations involving involution, including proving the existence of control function in two- and $n$-dimensional domains.

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## О задаче граничного управления для псевдопараболического уравнения с инволюцией

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Ранее были решены некоторые задачи управления псевдопараболическим уравнением, не зависящие от инволюции. В данной работе описывается задача граничного управления, связанная с псевдопараболическим уравнением с инволюцией в ограниченной одномерной области. На части границы изучаемой области дано значение решения с функцией управления. Ограничения на управление задаются таким образом, чтобы в рассматриваемой области среднее значение решения стало заданным. Задача, определяемая методом разделения переменных, сводится к интегральному уравнению Вольтерра второго рода. Существование функции управления доказано методом преобразования Лапласа.

*Ключевые слова*: краевая задача, интегральное уравнение Вольтерра, функция управления, преобразование Лапласа, инволюция.

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