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The methods to solve some classes boundary value problems via the results in double controlled metric-like spaces

N. Souayah[1,2], *Z. D. Mitrović*[3]

[1] King Saud University, P. Box 145111, Riyadh, 4545, Saudi Arabia
[2] University of Tunis, 92, Boulevard du 9 Avril 1938, Tunis, 1938-1007, Tunisia
[3] University of Banja Luka, 1A, Bulevar vojvode Petra Bojovica, Banja Luka, 78000, Bosnia and Herzegovina

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In this paper, we present some methods to solve second-order and fourth-order boundary value problems. First, we start by proving some new fixed point theorems in double controlled metric-like space. Further, we introduce the notion of G_ζ-contraction in the same space endowed with a graph and obtain a result on fixed points for G_ζ-contraction. As an application of the obtained results, we implemented the existence of solutions for some classes of second-order and fourth-order boundary value problems.

Keywords: differential equations, fixed point, graph theory, double controlled metric-like spaces.

1. Introduction. Due to its wide applicability, fixed point theory has received significant attention from researchers. Indeed, fixed point theorems are powerful tools in many areas of mathematics, telecommunication, physics, chemical graph theory, population biology [1–6]. The success of this tool has even reached new disciplines such as machine learning [7]. On the other hand, fixed point theorems offer a powerful method to ensure the existence of a solution to differential, integral and fractional differential equations under certain conditions [6, 8–15]. In the same context, the notion of metric spaces has been developed and generalized by many authors [13, 14, 16–26]. In [27], the authors proposed a new generalization of metric spaces called the double controlled metric spaces. Recently, Mlaiki [28], developed an extension named double controlled metric-like spaces (DCMLS), by supposing that the "self-distance" may not be zero.

In this article, we present selected applications of fixed point theory to prove the existence of a solution to certain types of differential equations. Also, we propose a fixed point result in DCMLS with a graph.

The rest of the paper is structured as follows. In Section 2 we state some definitions and results regarding double controlled metric-like spaces. In Section 3 we obtain the new fixed point results on double controlled metric-like spaces for an α-admissible maps (see Samet et al. [29]). In Section 4, using the ideas of Jachymski in [30], we give new results about fixed points on double controlled metric-like endowed with a graph. Also, we endowed the double controlled metric-like spaces by a graph G. Hence, the graph G is considered as weighted graph when the distance between its vertices is calculated by the double controlled metric-like. We introduce the definition of called the G_ζ-mapping. We establish a necessary conditions for this mapping and the controlled functions of the metric in order to prove existence and uniqueness of the fixed point (see Theorem 2 on p. 62). In Section 5, inspired by the results of Jleli and Samet [31], we present the fixed point results on double controlled metric-like for Θ-contractions. In Section 6, we discuss the existence of a solution of second order differential equation. Next, in Section 7, using our results, we solve a fourth-order differential equation.

2. Preliminaries. We begin by listing some fundamental definitions and properties of the double controlled metric-like spaces.

Defintion 1 [28]**.** Let X be a nonempty set. Let $\vartheta_1,\ \vartheta_2 : X\times X\to[1,+\infty)$ be the mappings and let $d : X\times X\to[0,+\infty)$ be a function that satisfies the following assumptions:

(dc_1) $d(p,q)=0$ implies $p=q$,

(dc_2) $d(p,q)=d(q,p)$,

(dc_3) $d(p,q)\leqslant\vartheta_1(p,r)d(p,r)+\vartheta_2(r,q)d(r,q)$,

for all $p,q,r\in X$.

The pair (X,d) is called a double controlled metric-like space DCMLS, and the function d is called a DCML.

The definition and convergence of Cauchy sequences in DCMLS is presented by the definition.

Defintion 2 [28]**.** Let (X,d) be a DCMLS and $\{r_n\}_{n\geqslant0}$ be a sequence in X:

(1) the sequence $\{r_n\}$ converges to some r in X, if and only if $\lim\limits_{n\to+\infty}d(r_n,r)=d(r,r)$;

(2) the sequence $\{r_n\}$ is Cauchy, if $\lim\limits_{n,m\to+\infty}d(r_n,r_m)$ exists and is finite;

(3) the space (X,d) is complete if every Cauchy sequence $\{r_n\}$ in X is convergent that is

$$\lim_{n\to+\infty}d(r_n,r)=d(r,r)=\lim_{n,m\to+\infty}d(r_n,r_m).$$

Defintion 3. Let (X,d) be a DCMLS. Let $r\in X$ and $\delta>0$:

(i) the open ball $\mathcal{B}(r,\delta)$ defined by

$$\mathcal{B}(r,\delta)=\{s\in X,|d(r,s)-d(r,r)|<\delta\};$$

(ii) the mapping $h : X\to X$ is called continuous at $r\in X$, if for each $\delta>0$ there exists $\gamma>0$ such that $h(\mathcal{B}_p(r,\gamma))\subseteq B_p(hr,\delta)$.

Clearly, if a mapping h is continuous at u in the space (X,d), then $u_n\to u$ implies that $hu_n\to hu$ as $n\to+\infty$, that is

$$\lim_{n\to+\infty}d(hu_n,hu)=d(hu,hu).$$

For more details and examples about the DCMLS, we refer to [28].

Remark 1. Note that Abdeljawad et al. [27] gave examples showing that doubled controlled metric space is a true generalization of controlled metric, b-metric and metric spaces (see examples in [28]).

3. Fixed point result in double controlled metric-like spaces for an α-admissible map. In this Section, we state a new fixed point result in the DCMLS and next, in Section 6, we will use the obtained result to solve a second-order differential equations. In the sequel, we state the following definitions.

Definition 4. Let $\gamma : X\to X$ be a map and let $\alpha : X\times X\to[0,+\infty)$. We say that γ is α-admissible if, for all $r,s\in X$, $\alpha(r,s)\geqslant1$ implies $\alpha(\gamma r,\gamma s)\geqslant1$.

Definition 5 [32]**.** Let $\gamma : X \to X$ be a map and let $\alpha, \kappa : X \times X \to [0, +\infty)$. We say that γ is α-admissible with respect to κ if, for all $r, s \in X$, $\alpha(r, s) \geqslant \kappa(r, s)$ implies $\alpha(\gamma r, \gamma s) \geqslant \kappa(\gamma r, \gamma s)$.

We denote by Ξ a new family of mappings $\chi : [0, +\infty) \to [0, +\infty)$ satisfying these conditions:

(Ξ_1) χ is an upper semicontinuous mapping from the right;

(Ξ_2) $\chi(r) < r$ for all $r \in (0, +\infty)$;

(Ξ_3) $\chi(0) = 0$.

Example. Let the maps $\chi_1, \chi_2, \chi_3 : [0, +\infty) \to [0, +\infty)$ defined by:

$\chi_1(t) = kt,\ t \in [0, +\infty)$, where $k \in (0, 1)$;

$\chi_2(t) = \ln(1 + t),\ t \in [0, +\infty)$;

$\chi_3(t) = \sin(t),\ t \in [0, +\infty)$. Then $\chi_1, \chi_2, \chi_3 \in \Xi$.

Lemma. *Let $\psi \in \Xi$ and $t \in (0, +\infty)$ then $\lim_{n\to+\infty} \psi^n(t) = 0$, where ψ^n is the n-th iterate of ψ.*

P r o o f. From condition (Ξ_2) we obtain

$$\psi^{n+1}(t) < \psi^n(t),\ \text{for all } n \in \mathbb{N}.$$

So, the sequence $\{\psi^n(t)\}$ is decreasing. In other hand, we have

$$\psi^n(t) \geqslant 0,\ \text{for all } n \in \mathbb{N}.$$

Therefore, we conclude that there exists $r \in [0, +\infty)$ such that

$$r = \lim_{n\to+\infty} \psi^n(t).$$

If $r > 0$, then from conditions (Ξ_1) and (Ξ_2) we obtain,

$$r \leqslant \limsup_{n\to+\infty} \psi(\psi^n(t)) \leqslant \psi(r) < r.$$

It is contradiction, so, $r = 0$. □

Theorem 1. *Let (X, d) be a complete DCMLS and $\psi \in \Xi$. Suppose that $f : X \to X$ is a continuous mapping and the following hypothesis hold:*

(i) f is α-admissible with respect to κ;

(ii) if $r, s \in X$ and $\alpha(r, s) \geqslant \kappa(r, s)$, then $d(fr, fs) \leqslant \psi(d(r, s))$;

(iii) there exists $r_0 \in X$ such that $\alpha(r_0, fr_0) \geqslant \kappa(r_0, fr_0)$.

We take $r_n = \zeta^n r_0$ and we suppose that we have

$$\lim_{i\to+\infty} \vartheta_1(r_i, r_{i+1}) \text{ and } \lim_{i\to+\infty} \vartheta_2(r_i, r_m) \text{ exist and finite.} \tag{1}$$

Then f has a fixed point.

P r o o f. Let r_0 be an element in X. We denote by $r_1 = fr_0$ and we construct the following sequence $\{r_n\} \in X$ defined by

$$r_{n+1} = fr_n,\ \text{for all } n \in \mathbb{N}.$$

Suppose that $r_n \neq r_{n+1}$ for all $n \in \mathbb{N}$, otherwise f has a fixed point. From condition (ii) we have

$$\alpha(r_0, r_1) = \alpha(r_0, fr_0) \geqslant \kappa(r_0, fr_0)$$

and taking in to account that f is α-admissible with respect to κ we obtain that

$$\alpha(r_1, r_2) = \alpha(fr_0, fr_1) \geqslant \kappa(fr_0, fr_1) = \kappa(r_1, r_2).$$

By continuing this process, we obtain

$$\alpha(r_n, r_{n+1}) \geqslant \kappa(r_n, r_{n+1}), \text{ for all } n \in \mathbb{N}.$$

From (ii) and the property of ψ, we obtain that

$$d(r_n, r_{n+1}) = d(fr_{n-1}, fr_n) \leqslant \psi(d(r_{n-1}, r_n)) < d(r_{n-1}, r_n), \text{ for all } n \in \mathbb{N}. \tag{2}$$

Therefore, $\{d(r_n, r_{n+1})\}$ is a non-increasing sequence. By consequence, there exists $r \geqslant 0$ such that

$$\lim_{n\to+\infty} d(r_n, r_{n+1}) = r.$$

We claim that $r = 0$. Suppose that $r > 0$. Since ψ is upper semicontinuous from the right using (2), we get

$$\lim_{n\to+\infty} d(r_n, r_{n+1}) = r \leqslant \lim \sup_{n\to+\infty} \psi(d(r_{n-1}, r_n)) \leqslant \psi(r) < r,$$

which is a contradiction. Therefore,

$$\lim_{n\to+\infty} d(r_n, r_{n+1}) = 0. \tag{3}$$

Let $i, j \in \mathbb{N}$ such that $r_i \neq r_j$ for all $i \neq j$. Suppose without loss of generality that $i < j$. Using the triangle inequality of the DCML, we get

$$\begin{aligned} d(r_i, r_j) &\leqslant \vartheta_1(r_i, r_{i+1})d(r_i, r_{i+1}) + \vartheta_2(r_{i+1}, r_j)d(r_{i+1}, r_j) \leqslant \\ &\leqslant \vartheta_1(r_i, r_{i+1})d(r_i, r_{i+1}) + \vartheta_2(r_{i+1}, r_j)\vartheta_1(r_{i+1}, r_{i+2})d(r_{i+1}, r_{i+2}) + \\ &+ \vartheta_2(r_{i+1}, r_j)\vartheta_2(r_{i+2}, r_j)d(r_{i+2}, r_j) \leqslant \\ &\vdots \\ &\leqslant \vartheta_1(r_i, r_{i+1})d(r_i, r_{i+1}) + \sum_{p=i+1}^{j-2} \left(\prod_{q=i+1}^{p} \vartheta_2(r_q, r_j) \right) \times \\ &\times \vartheta_1(r_p, r_{p+1})d(r_p, r_{p+1}) + \prod_{z=i+1}^{j-1} \vartheta_2(r_z, r_j)d(r_{j-1}, r_j). \end{aligned}$$

Using (1) and (3) we get that $d(r_i, r_j)$ converges to 0 as $i, j \to +\infty$.

Thus, $\{r_n\}$ is a Cauchy sequence and then converges to some $r \in X$. Due to the continuity of f, we have

$$r = \lim_{n\to+\infty} r_{n_l+1} = \lim_{n\to+\infty} fr_{n_l} = fr.$$

Hence, r is a fixed point of f. □

Remark 2. It is known that every DCMLS is a metric space. But the converse is not always true which allows us to conclude that the DCMLS is more general than metric spaces. Given this fact, it follows that the previous theorem is a generalization of some fixed point results obtained in metric spaces. We can cite the Theorem 12 in [33]. Indeed, in [33], the authors prove the existence of the fixed point in a complete metric spaces for α-admissible mappings using the same contraction in our result.

4. Fixed point result in double controlled metric-like spaces endowed with a graph. Now, we combine the fixed point and graph theory to provide our result. We start by giving some basic notions of graph theory needed hereafter. According to Jachymski in [30], we consider a DCMLS (X, d) endowed with a graph G defined by the set $U = U(G)$ of vertices coinciding with X and the set $E = E(G)$ of its edges. Assume that the graph G has no parallel edges. Therefore, G can be identified with the pair (U, E). Moreover, the graph G may be seen as a weighted graph. Indeed, the distance between vertices is calculated by the d-double controlled metric-like considered as a weight for the associated edge. For more details in graph theory, we refer to [30, 34].

Definition 6. Let (X, d) be a complete (DCMLS) with a graph G. The mapping ζ is called a G_ζ-mapping if the following hypothesis hold:

(1) for all $r, s \in E(G)$, $(r, s) \in E(G)$ imply that $(\zeta r, \zeta s) \in E(G)$, ($G$-edge preserving);

(2) there exists a function $\varphi : X \to \mathbb{R}$ bounded from below such that

$$d(\zeta r, \zeta s) \leqslant (\varphi(r) - \varphi(\zeta r))d(r, s), \tag{4}$$

for all $(\zeta r, \zeta s) \in E(G)$ or $(r, s) \in E(G)$.

Theorem 2. *Let (X, d, G) be a complete (DCMLS) endowed with a graph G. Let $\zeta : X \to X$ be a continuous G_ζ-mapping. Assume that there exists $r_0 \in X$ such that*

$$(\zeta r_0, r_0) \in E(G). \tag{5}$$

We take $r_n = \zeta^n r_0$ and we suppose that for every $r \in X$ we have

$$\lim_{i\to+\infty} \vartheta_1(r_i, r) \text{ and } \lim_{i\to+\infty} \vartheta_2(r_i, r_{i+1}) \text{ exist and finite.}$$

Then ζ has a unique fixed point.

P r o o f. The equation (5) implies that there exists $r_0 \in X$ such that $(\zeta r_0, r_0) \in E(G)$. Since ζ is G-edge preserving, we obtain

$$(\zeta^{n+1} r_0, \zeta^n r_0) \in E(G), \text{ for all } n \in \mathbb{N}.$$

Let $r_0 \in X$, we assume that $d(\zeta r_0, r_0) > 0$ otherwise the proof is completed. Subsequently, $d(r_n, r_{n+1}) = d(r_n, \zeta r_n) > 0$. Let it be

$$\alpha_n = d(r_{n-1}, r_n).$$

From (4) we obtain

$$\alpha_{n+1} = d(r_n, r_{n+1}) = d(\zeta r_{n-1}, \zeta r_n) \leqslant (\varphi(r_{n-1}) - \varphi(\zeta r_{n-1}))d(r_{n-1}, r_n) =$$
$$= (\varphi(r_{n-1}) - \varphi(r_n))\alpha_n.$$

Hence,

$$0 < \frac{\alpha_{n+1}}{\alpha_n} \leqslant \varphi(r_{n-1}) - \varphi(r_n), \text{ for all } n \in \mathbb{N}. \tag{6}$$

Therefore, the sequence $(\varphi(r_n))$ is non-increasing and positive. Thereby, $\lim_{n\to+\infty} \varphi(r_n) = r > 0$. Now, using (6) we get

$$\sum_{i=1}^{n} \frac{\alpha_{i+1}}{\alpha_i} \leqslant \sum_{i=1}^{n} (\varphi(r_{i-1}) - \varphi(r_i)) = \varphi(r_0) - \varphi(r_1) + \varphi(r_1) - \cdots + \varphi(r_{n-1}) - \varphi(r_n) =$$
$$= \varphi(r_0) - \varphi(r_n),$$

which means that $\sum_{i=1}^{+\infty} \frac{\alpha_{i+1}}{\alpha_i} < +\infty$. Consequently, we have

$$\lim_{i\to+\infty} \frac{\alpha_{i+1}}{\alpha_i} = 0. \tag{7}$$

Taking into account (7), there exists $i_0 \in \mathbb{N}$ such that for all $i \geqslant i_0$

$$\frac{\alpha_{i+1}}{\alpha_i} \leqslant K, \quad \text{for } K \in (0,1).$$

It yields that

$$d(r_{i+1}, r_i) \leqslant K d(r_i, r_{i-1}), \text{ for all } i \geqslant i_0. \tag{8}$$

Now, we show that $\{r_i\}$ is a Cauchy sequence. From (8), we get

$$d(r_{i+1}, r_i) \leqslant K^i d(r_0, r_1), \text{ for all } i \geqslant i_0. \tag{9}$$

On the other hand, from (4),

$$d(\zeta^n r_0, \zeta^{n+1} r_0) \leqslant (\varphi(\zeta^{n-1} r_0) - \varphi(\zeta^n r_0)) d(\zeta^{n-1} r_0, \zeta^n r_0),$$

which gives $\varphi(\zeta^{n-1} r_0) - \varphi(\zeta^n r_0) \geqslant 0$. Hence $\{\varphi(\zeta^n r_0)\}$ is a decreasing sequence of positive numbers. Let $\varphi_0 = \lim_{n\to+\infty} \varphi(\zeta^n r_0)$.

For any $n, m \in \mathbb{N}$, we have

$$\begin{aligned}
d(\zeta^n r_0, \zeta^{n+m} r_0) &\leqslant (\varphi(\zeta^{n-1} r_0) - \varphi(\zeta^n r_0)) d(\zeta^{n-1} r_0, \zeta^{n+m-1} r_0) \leqslant \\
&\leqslant (\varphi(\zeta^{n-1} r_0) - \varphi(\zeta^n r_0))(\varphi(\zeta^{n-2} r_0) - \varphi(\zeta^{n-1} r_0)) d(\zeta^{n-2} r_0, \zeta^{n+m-2} r_0) \leqslant \\
&\vdots \\
&\leqslant \prod_{k=1}^{n} \left(\varphi(\zeta^{n-k} r_0) - \varphi(\zeta^{n-k+1} r_0)\right) d(\zeta^0 r_0, \zeta^m r_0).
\end{aligned}$$

From the properties of φ we have

$$\prod_{k=1}^{n} (\varphi(\zeta^{n-k} r_0) - \varphi(\zeta^{n-k+1} r_0)) \to 0 \text{ as } n \to +\infty.$$

We claim that

$$\lim_{m\to+\infty} d(\zeta^0 r_0, \zeta^m r_0) = 0.$$

Using the condition (dc_3) in Definition 1 we get

$$\begin{aligned}
d(\zeta^0 r_0, \zeta^m r_0) = d(r_0, r_m) &\leqslant \\
&\leqslant \vartheta_1(r_0, r_1) d(r_0, r_1) + \vartheta_2(r_1, r_m) d(r_1, r_m) \leqslant \\
&\leqslant \vartheta_1(r_0, r_1) d(r_0, r_1) + \vartheta_2(r_1, r_m) \vartheta_1(r_1, r_2) d(r_1, r_2) + \\
&+ \ \vartheta_2(r_1, r_m) \vartheta_2(r_2, r_m) d(r_2, r_m) \leqslant \\
&\vdots
\end{aligned}$$

$$\leqslant \vartheta_1(r_0,r_1)d(r_0,r_1) + \sum_{p=1}^{m-2}\left(\prod_{q=1}^{p}\vartheta_2(r_q,r_m)\right)\times$$
$$\times\,\vartheta_1(r_p,r_{p+1})d(r_p,r_{p+1}) + \prod_{k=1}^{m-1}\vartheta_2(r_k,r_m)d(r_{m-1},r_m).$$

Using (9) we obtain

$$\begin{aligned}
d(\zeta^0 r_0,\zeta^m r_0) &\leqslant \vartheta_1(r_0,r_1)d(r_0,r_1) + \sum_{p=1}^{m-2}\left(\prod_{q=1}^{p}\vartheta_2(r_q,r_m)\right)\times\\
&\times\,\vartheta_1(r_p,r_{p+1})K^p d(r_0,r_1) + \prod_{k=1}^{m-1}\vartheta_2(r_k,r_m)K^{m-1}d(r_0,r_m) \leqslant\\
&\leqslant \vartheta_1(r_0,r_1)d(r_0,r_1) + \sum_{p=1}^{m-1}\left(\prod_{q=1}^{p}\vartheta_2(r_q,r_m)\right)\times\\
&\times\,\vartheta_1(r_p,r_{p+1})K^p d(r_0,r_1) \leqslant \vartheta_1(r_0,r_m)\vartheta_2(r_0,r_1)d(r_0,r_1)\,+\\
&+\sum_{p=1}^{m-1}\left(\prod_{q=1}^{p}\vartheta_2(r_q,r_m)\right)\vartheta_1(r_p,r_{p+1})K^p d(r_0,r_1) \leqslant\\
&\leqslant \sum_{p=0}^{m-1}\left(\prod_{q=0}^{p}\vartheta_2(r_q,r_m)\right)\vartheta_1(r_p,r_{p+1})K^p d(r_0,r_1) =\\
&= \sum_{p=0}^{m-1}\gamma_p d(r_0,r_1),
\end{aligned}$$

where

$$\gamma_p = \left(\prod_{q=0}^{p}\vartheta_2(r_q,r_m)\right)\vartheta_1(r_p,r_{p+1})K^p.$$

From the properties of the controlled functions ϑ_1, ϑ_2 and the constant K we can deduce that

$$\lim_{m\to+\infty} d(\zeta^0 r_0,\zeta^m r_0) = 0.$$

Thereafter, $\lim_{m\to+\infty} d(\zeta^n r_0,\zeta^{n+m} r_0) = 0$. Thus $\{\zeta^n r_0\}$ is a Cauchy sequence in the space X. Then, there exists $r^*\in X$ such that $\{\zeta^n r_0\}$ converges to r^* as $n\to+\infty$. Due to the continuity of ζ we obtain that $\zeta r^* = r^*$. Therefore, r^* is a fixed point of ζ.

Assume there would be two different fixed points r_1^* and r_2^* in X such that $\zeta r_1^* = r_1^*$ and $\zeta r_2^* = r_2^*$. We have

$$d(r_1^*,r_2^*) = d(\zeta r_1^*,\zeta r_2^*) \leqslant (\varphi(r_1^*)-\varphi(\zeta r_1^*))d(r_1^*,r_2^*) = (\varphi(r_1^*)-\varphi(r_1^*))d(r_1^*,r_2^*) = 0,$$

then $d(r_1^*,r_2^*)=0$ implies $u_1^* = v_1^*$. □

Remark 3. The Theorem 2 is a generalization of Theorem 1 in [35]. In fact, in [35] the authors present a Caristi type fixed point in b-metric spaces. It is clear that the considered metric in the above result that is DCMLS endowed with graph is more general than the b-metric space.

5. Fixed point result in double controlled metric-like spaces for Θ-contractions. In this section, we present our second result inspired from the recent work of Jleli and Samet [31]. Next, we use the fixed point theorem to establish the existence of the solution of a boundary value problem and check its uniqueness. Recently, authors [31] proposed a concept of Θ-contraction. We start by defining the set $\Theta := \{\breve{\theta} : (0,+\infty) \to (1,+\infty)\}$ as follows:

($\star$) $\breve{\theta}$ is continuous and non-decreasing;

($\star\star$) for each sequence $\{\beta_n\} \subset (0,+\infty)$, $\lim\limits_{n\to+\infty} \breve{\theta}(\beta_n) = 1 \Leftrightarrow \lim\limits_{n\to+\infty} \beta_n = 0^+$;

($\star\star\star$) there exist $\kappa \in (0,1)$ and $l_{\breve{\theta}} \in (0,+\infty]$ such that $\lim\limits_{\beta\to 0^+} \dfrac{\breve{\theta}(\beta)-1}{\beta^\kappa} = l_{\breve{\theta}}$.

Theorem 3. *Let (X,d) be a DCMLS and $\zeta : X \to X$ be a continuous mapping satisfying the following condition: there exists a function $\breve{\theta} \in \Theta$ such that*

$$\breve{\theta}(d(\zeta u, \zeta v)) \leqslant [\breve{\theta}(d(u,v))]^r, \quad \text{if } \ d(\zeta u, \zeta v) \neq 0 \text{ for } u, v \in X, \tag{10}$$

where $r \in (0,1)$. Moreover, we take $r_n = \zeta^n r_0$ for $r_0 \in X$ and we assume that for every $r \in X$ we have

$$\lim_{i\to+\infty} \vartheta_1(r_i, r_{i+1}) \text{ and } \lim_{i\to+\infty} \vartheta_2(r_i, r_m) \text{ exist and finite.} \tag{11}$$

Then ζ has a unique fixed point in X.

P r o o f. Let fix an arbitrary point $r_0 \in X$. We build an iterative sequence $\{r_n\}$ as follows:

$$r_n = \zeta^n r_0, \text{ for all } n \in \mathbb{N}.$$

Suppose, if $r_{n^*} = r_{n^*+1}$ for some $n^* \in \mathbb{N}$, then r_{n^*} is a trivial fixed point of ζ.

Thus, we suppose, without loss of generality, that $d(r_n, r_{n+1}) > 0$ for all $n \in \mathbb{N}$. From (10), we have

$$\breve{\theta}(d(r_n, r_{n+1})) = \breve{\theta}(d(\zeta r_{n-1}, \zeta r_n)) \leqslant [\breve{\theta}(d(r_{n-1}, r_n))]^r \leqslant [\breve{\theta}(d(r_{n-2}, r_{n-1}))]^{r^2}.$$

By continuing this process, we get

$$1 < \breve{\theta}(d(r_n, r_{n+1})) \leqslant [\breve{\theta}(d(r_0, r_1))]^{r^n}, \text{ for all } n \in \mathbb{N}. \tag{12}$$

Letting $n \to +\infty$ in (12), we get $\breve{\theta}(d(r_n, r_{n+1})) \to 1$ as $n \to +\infty$.

From ($\star\star$), we have

$$\lim_{n\to+\infty} d(r_n, r_{n+1}) = 0.$$

Similarly, we can easily obtain that

$$\lim_{n\to+\infty} d(r_n, r_{n+2}) = 0.$$

From ($\star\star\star$), there exist $\kappa \in (0,1)$ and $l_{\breve{\theta}} \in (0,+\infty]$ such that

$$\lim_{n\to+\infty} \frac{\breve{\theta}(d(r_n, r_{n+1})) - 1}{[d(r_n, r_{n+1})]^\kappa} = l_{\breve{\theta}}.$$

Suppose that $l_{\breve{\theta}} < +\infty$. In this case, let $A = \dfrac{l_{\breve{\theta}}}{2} > 0$. Using limit definition, we pick $n_0 \in \mathbb{N}$ such that

$$\left| \frac{\breve{\theta}(d(r_n, r_{n+1})) - 1}{[d(r_n, r_{n+1})]^\kappa} - l_{\breve{\theta}} \right| \leqslant A, \text{ for all } n \geqslant n_0.$$

This implicates that $\left|\dfrac{\check{\theta}(d(r_n, r_{n+1}))-1}{[d(r_n, r_{n+1})]^\kappa}\right| \geqslant l_{\check{\theta}} - A = A$, for all $n \geqslant n_0$.

Then, we infer that

$$n\left[d(r_n, r_{n+1})^\kappa\right] \leqslant n\left[\frac{\check{\theta}(d(r_n, r_{n+1}))-1}{A}\right], \text{ for all } n \geqslant n_0.$$

Suppose that $l_{\check{\theta}} = +\infty$. Let $A > 0$ be an arbitrary positive number. Using the limit definition, we find $n_0 \in \mathbb{N}$ such that

$$\frac{\check{\theta}(d(r_n, r_{n+1}))-1}{[d(r_n, r_{n+1})]^\kappa} \geqslant A, \text{ for all } n \geqslant n_0.$$

This implies that

$$n[d(r_n, r_{n+1})]^\kappa \leqslant n\left[\frac{\check{\theta}(d(r_n, r_{n+1}))-1}{A}\right], \text{ for all } n \geqslant n_0.$$

Thus, in all cases, there exist $\frac{1}{A} > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[d(r_n, r_{n+1})]^\kappa \leqslant n\left[\frac{\check{\theta}(d(r_n, r_{n+1}))-1}{A}\right], \text{ for all } n \geqslant n_0.$$

Using equation (12), we obtain

$$n[d(r_n, r_{n+1})]^\kappa \leqslant [\check{\theta}(d(r_0, r_1))]^{r^n} - 1, \text{ for all } n \geqslant n_0. \tag{13}$$

If we let $n \to +\infty$ in (13), then we get

$$\lim_{n\to+\infty} n[d(r_n, r_{n+1})]^\kappa = 0.$$

Hence, we can find $n_1 \in \mathbb{N}$ such that

$$d(r_n, r_{n+1}) \leqslant \frac{1}{n^{\frac{1}{\kappa}}}, \tag{14}$$

for all $n \geqslant n_1$. Let $i, j \in \mathbb{N}$ such that $r_i \neq r_j$ for all $i \neq j$, otherwise it is easy to conclude that r_i is a fixed point of ζ. Suppose without loss of generality that $i < j$. Using the triangle inequality of the DCML, we get

$$\begin{aligned}
d(r_i, r_j) &\leqslant \vartheta_1(r_i, r_{i+1})d(r_i, r_{i+1}) + \vartheta_2(r_{i+1}, r_j)d(r_{i+1}, r_j) \leqslant \\
&\leqslant \vartheta_1(r_i, r_{i+1})d(r_i, r_{i+1}) + \vartheta_2(r_{i+1}, r_j)\vartheta_1(r_{i+1}, r_{i+2})d(r_{i+1}, r_{i+2}) + \\
&+\ \vartheta_2(r_{i+1}, r_j)\vartheta_2(r_{i+2}, r_j)d(r_{i+2}, r_j) \leqslant \\
&\ \vdots \\
&\leqslant \vartheta_1(r_i, r_{i+1})d(r_i, r_{i+1}) + \sum_{p=i+1}^{j-2}\left(\prod_{q=i+1}^{p}\vartheta_2(r_q, r_j)\right) \times \\
&\times \vartheta_1(r_p, r_{p+1})d(r_p, r_{p+1}) + \prod_{z=i+1}^{j-1}\vartheta_2(r_z, r_j)d(r_{j-1}, r_j).
\end{aligned}$$

Using (14) we get

$$d(r_i,r_j) \leqslant \vartheta_1(r_i,r_{i+1})\frac{1}{i^{\frac{1}{\kappa}}} + \sum_{p=i+1}^{j-2}\left(\prod_{q=i+1}^{p}\vartheta_2(r_q,r_j)\right)\vartheta_1(r_p,r_{p+1})\frac{1}{p^{\frac{1}{\kappa}}} +$$
$$+ \prod_{z=i+1}^{j-1}\vartheta_2(r_z,r_j)\frac{1}{j^{\frac{1}{\kappa}}} \leqslant$$
$$\leqslant \vartheta_1(r_i,r_{i+1})\vartheta_2(r_i,r_j)\frac{1}{i^{\frac{1}{\kappa}}} + \sum_{p=i+1}^{j-2}\left(\prod_{q=i+1}^{p}\vartheta_2(r_q,r_j)\right)\times$$
$$\times\,\vartheta_1(r_p,r_{p+1})\frac{1}{p^{\frac{1}{\kappa}}} + \prod_{z=i+1}^{j-1}\vartheta_2(r_z,r_j)\frac{1}{j^{\frac{1}{\kappa}}} \leqslant$$
$$\leqslant \sum_{p=i}^{j-2}\left(\prod_{q=i}^{p}\vartheta_2(r_q,r_j)\right)\vartheta_1(r_p,r_{p+1})\frac{1}{p^{\frac{1}{\kappa}}} + \prod_{z=i+1}^{j-1}\vartheta_2(r_z,r_j)\frac{1}{j^{\frac{1}{\kappa}}} \leqslant$$
$$\leqslant \sum_{p=i}^{j-1}\left(\prod_{q=i}^{p}\vartheta_2(r_q,r_j)\right)\vartheta_1(r_p,r_{p+1})\frac{1}{p^{\frac{1}{\kappa}}}.$$

Therefore, due to the properties of ϑ_1 and ϑ_2 given by (11) and knowing that $\frac{1}{\kappa} \geqslant 1$, we obtain that

$$d(r_i,r_j) \to 0 \text{ as } i,j \to +\infty.$$

Hence, the sequence $\{r_i\}$ is a Cauchy sequence in X. Since (X,d) is a complete DCMLS, there exists a point r in X such that $\lim\limits_{i\to+\infty} r_i = r$. By using the continuity of ζ we can conclude that $= r$. Thus, ζ has a fixed point. Suppose that ζ has two different fixed points r_1 and r_2. Therefore, using (10), we obtain

$$\check{\theta}(d(r_1,r_1)) = \check{\theta}(d(\zeta r_1,\zeta r_2)) = \check{\theta}(d(\zeta^\kappa r_1,\zeta^\kappa r_2)) \leqslant [\check{\theta}(d(r_1,r_2))]^{r^\kappa} < \check{\theta}(d(r_1,r_2)),$$

which is a contradiction. Hence, $r_1 = r_2$. □

6. Resolution of second-order differential equation. Now, we use our obtained result in Theorem 1 to verify the existence of a solution to the following problem $(\mathcal{R})$:

$$\begin{aligned} -\frac{d^2w}{dr^2} &= h(r,w(r)), \quad r\in[0,1], \\ w(0) &= w(1) = 0, \end{aligned} \tag{15}$$

where $h : [0,1]\times\mathbb{R} \to \mathbb{R}$ is continuous. The Green function related to (15) is defined by

$$G(r,s) = \begin{cases} r(1-s), & 0 \leqslant r \leqslant s \leqslant 1, \\ s(1-r), & 0 \leqslant s \leqslant r \leqslant 1. \end{cases}$$

Denote by

$$\mathcal{C}([0,1]) = \{\xi : [0,1] \to [0,1] : \xi \text{ is continuous}\}.$$

Let $d : \mathcal{C}([0,1])^2 \to \mathbb{R}$ be defined by

$$d(r,s) = ||r-s||_\infty = \max_{k\in[0,1]} |r(k)-s(k)|.$$

It is easy to see that $(C([0,1],d)$ is a DCMLS.

Theorem 4. *Consider the problem* $(\mathcal{R})$*. Assume that the following assumptions hold:*

(1) there exists a function $\Phi : \mathbb{R}^2 \to \mathbb{R}$ *such that, for all* $r \in [0,1]$*, and* $b_1, b_2 \in \mathbb{R}$ *with* $\Phi(b_1, b_2) \geqslant 0$*, we have*

$$|h(r,b_1) - h(r,b_2)| \leqslant 8\psi\left(\max_{b_1,b_2\in\mathbb{R},\Phi(b_1,b_2)\geqslant 0} |b_1 - b_2|\right);$$

(2) there exists $w_0 \in \mathcal{C}([0,1])$ *such that, for all* $r \in [0,1]$*, we have*

$$\Phi\Big(w_0(r), \int_0^1 G(r,s)h(s,w_0(s))ds\Big) \geqslant 0;$$

(3) if $\{w_n\}$ *is a sequence in* $\mathcal{C}([0,1])$ *such that* $w_n \to w \in \mathcal{C}([0,1])$ *and* $\Phi(w_n, w_{n+1}) \geqslant 0$*,* $\forall n \in \mathbb{N}$*, then* $\Phi(w_n, w) \geqslant 0$*, for all* $n \in \mathbb{N}$*;*

(4) for all $r \in [0,1]$*, for all* $w, \rho \in \mathcal{C}([0,1])$*,* $\Phi(w(r), \rho(r)) \geqslant 0$ *implies*

$$\Phi\Big(\int_0^1 G(r,s)h(s,w(s))ds, \int_0^1 G(r,s)h(s,\rho(s))ds\Big) \geqslant 0.$$

Then, $(\mathcal{R})$ *has a solution in* $C^2([0,1])$*.*

P r o o f. Solving the problem $(\mathcal{R})$ is equivalent to solving the integral equation

$$w(r) = \int_0^r G(r,s)h(s,w(s))ds, \text{ for all } r \in [0,1].$$

Let f be a self-mapping on $\mathcal{C}([0,1])$ defined by

$$fw(r) = \int_0^1 G(r,s)h(s,w(s))ds, \text{ for all } r \in [0,1].$$

Suppose that $w, \rho \in \mathcal{C}([0,1])$ such that

$$\Phi(w(r), \rho(r)) \geqslant 0, \text{ for all } r \in [0,1].$$

Using the first assumption of the theorem, we obtain that

$$\begin{aligned}
|fw(r) - f\rho(r)| &= \Big|\int_0^1 G(r,s)[h(s,w(s)) - h(s,\rho(s))]ds\Big| \leqslant \\
&\leqslant \int_0^1 G(r,s)\Big|h(s,w(s)) - h(s,\rho(s))\Big|ds \leqslant \\
&\leqslant 8\Big(\int_0^1 G(r,s)ds\Big)(\psi(||w-\rho||_\infty)) \leqslant \\
&\leqslant 8\Big(\sup_{r\in[0,1]} \int_0^1 G(r,s)ds\Big)(\psi(||w-\rho||_\infty)).
\end{aligned}$$

As

$$\int_0^1 G(r,s)ds = -\frac{r^2}{2} + \frac{r}{2},$$

for all $r \in [0,1]$, we get

$$\sup_{r\in[0,1]} \int_0^1 G(r,s)ds = \frac{1}{8}.$$

Consequently

$$||fw - f\rho||_\infty \leqslant \psi(||w-\rho||_\infty),$$

for each $w, \rho \in \mathcal{C}([0,1])$, such that

$$\Phi(w(r), \rho(r)) \geqslant 0, \text{ for all } r \in [0,1].$$

Therefore, the condition (ii) of Theorem 1 is hold. Now, let us prove that f is α-admissible concerning κ. Let

$$\alpha, \kappa : \mathcal{C}([0,1])^2 \to [0, +\infty)$$

be the mappings defined by

$$\alpha(w,\rho) = \begin{cases} 1, & \Phi(w(r), \rho(r)) \geqslant 0, \quad r \in [0,1], \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\kappa(w,\rho) = \begin{cases} \dfrac{1}{2}, & \Phi(w(r), \rho(r)) \geqslant 0, \quad r \in [0,1], \\ 2, & \text{otherwise.} \end{cases}$$

Let $w, \rho \in \mathcal{C}([0,1])$ such that $\alpha(w,\rho) \geqslant \kappa(w,\rho)$. Hence,

$$\Phi(w(r), \rho(r)) \geqslant 0, \text{ for all } r \in [0,1].$$

Hence,

$$||fw - f\rho||_\infty \leqslant \psi(||w-\rho||_\infty).$$

Furthermore, if $w, \rho \in \mathcal{C}([0,1])$ such that $\alpha(w,\rho) \geqslant \kappa(w,\rho)$ then by applying the assumption (4) we obtain $\Phi(fw(r), f\rho(r)) \geqslant 0$ and this gives that $\alpha(fw, f\rho) \geqslant \kappa(fw, f\rho)$. Hence, f is α-admissible with respect to κ. Using the condition (2), there exists $w_0 \in \mathcal{C}([0,1])$ such that $\alpha(w_0, fw_0) \geqslant \kappa(w_0, fw_0)$. So, all the conditions of Theorem 1 are hold, thus f has are a fixed point in $\mathcal{C}([0,1])$ say w^* solution of the problem $(\mathcal{R})$. □

7. Resolution of fourth-order differential equation. In the sequel, we study the existence of solution of the following fourth-order differential equation boundary problem $(\mathcal{P})$ using the result given by Theorem 3:

$$\begin{cases} \zeta^4(t) = f(t, \zeta(t), \zeta', \zeta'', \zeta'''), \\ \zeta(0) = \zeta'(0) = \zeta''(1) = \zeta'''(1) = 0; \ t \in [0,1]. \end{cases}$$

We will use the Theorem 3 to prove the existence and uniqueness of the solution of the above problem. Let $X = \mathcal{C}[0,1]$, where

$$\mathcal{C}([0,1]) = \{\xi : [0,1] \to [0,1] : \xi \text{ is continuous}\}.$$

We consider the DCMLS on X^2 defined by

$$d(r_1, r_2) = |r_1 - r_2|^2.$$

It is easy to check that $(\mathcal{C}[0,1], d)$ is a DCMLS. Indeed, the conditions (dc_1) and (dc_2) of Definition 1 are satisfied. For the triangle inequality (dc_3), let $f, g, h \in \mathcal{C}[0,1]$ and ϑ_1, $\vartheta_2 : \mathcal{C}[0,1]^2 \to [2, +\infty)$, we have

$$\begin{aligned} d(f,g) = |f-g|^2 &= |f-h+h-g|^2 \leqslant \\ &\leqslant 2\left(|f-h|^2 + |h-g|^2\right) \leqslant \\ &\leqslant \vartheta_1(f,h)|f-h|^2 + \vartheta_2(h,g)|h-g|^2. \end{aligned}$$

Using the integral form, the problem $(\mathcal{P})$ can be written:

$$\zeta(t) = \int\limits_0^1 \mathbb{G}(u,v) f(v, \zeta(v), \zeta'(v)) dv, \quad \zeta \in \mathcal{C}[0,1],$$

where $\mathbb{G}(u,v)$ is the Green's function of the homogenous linear problem $\zeta^4(u) = 0$, $\zeta(0) = \zeta'(0) = \zeta''(1) = \zeta'''(1) = 0$, represented by

$$\mathbb{G}(u,v) = \begin{cases} \frac{1}{6}u^2(3v-u), & 0 \leqslant u \leqslant v \leqslant 1, \\ \frac{1}{6}v^2(3u-v), & 0 \leqslant v \leqslant u \leqslant 1. \end{cases} \tag{16}$$

From (16), we can affirm that $\mathbb{G}(u,v)$ has the properties

$$\frac{1}{3}u^2v^2 \leqslant \mathbb{G}(u,v) \leqslant \frac{1}{2}u^2 \text{ (or } \frac{1}{2}v^2), \quad u, v \in [0,1].$$

Theorem 5. *Assume that the following assumptions hold:*
(1) $h : [0,1] \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ *is continuous*;
(2) there exists $\tau \in [1, +\infty)$ *such that for all* $\zeta, z \in X$:

$$|h(v, \zeta, \zeta') - h(v, z, z')| \leqslant \sqrt{20} e^{\frac{-\tau}{2}} |\zeta(v) - z(v)|, \; v \in [0,1];$$

(3) there exists $\zeta_0 \in X$ *such that for all* $u \in [0,1]$, *we have*

$$\zeta_0 u \leqslant \int\limits_0^1 \mathbb{G}(u,v) h(v, \zeta_0(v), \zeta_0'(v)) dv.$$

Then the problem $(\mathcal{P})$ *has a solution in* X.

P r o o f. If we define the mapping $h : X \to X$ by

$$h(\zeta)(u) = \int\limits_0^1 \mathbb{G}(u,v) h(v, \zeta(v), \zeta'(v)) dv.$$

Then $\zeta = h\zeta$. Consequently, $(\mathcal{P})$ has a unique solution. Consider,

$$|h(\zeta)(u) - h(z)(u)|^2 = \Big| \int_0^1 \mathbb{G}(u,v)h(v,\zeta(v),\zeta'(v))dv - \int_0^1 \mathbb{G}(u,v)h(v,z(v),z'(v))dv \Big|^2 \leqslant$$

$$\leqslant \int_0^1 (\mathbb{G}(u,v))^2 |h(v,\zeta(v),\zeta'(v)) - h(v,z(v),z'(v))|^2 dv \leqslant$$

$$\leqslant \int_0^1 \frac{1}{4} v^4 20e^{-\tau} |\zeta(v) - z(v)|^2 dv \leqslant 20e^{-\tau} d(\zeta,z) \int_0^1 \frac{1}{4} v^4 dv \leqslant$$

$$\leqslant 20e^{-\tau} d(\zeta,z) \frac{1}{20} = e^{-\tau} d(\zeta,z)$$

which yields,

$$d(h(\zeta),h(z)) \leqslant e^{-\tau} d(\zeta,z),$$
$$\sqrt{d(h(\zeta),h(z))} \leqslant \sqrt{e^{-\tau} d(\zeta,z)},$$
$$e^{\sqrt{d(h(\zeta),h(z))}} \leqslant \left(e^{\sqrt{d(\zeta,z)}}\right)^{\sqrt{e^{-\tau}}},$$

here $e^{-\tau} < 1$ as $\tau \geqslant 1$. Hence

$$e^{\sqrt{d(h(\zeta),h(z))}} \leqslant \left(e^{\sqrt{d(\zeta,z)}}\right)^{\sqrt{r}}$$

with $r = \sqrt{e^{-\tau}}$ which gives,

$$\check{\theta}(d(h\zeta,hz)) \leqslant [\check{\theta}(d(\zeta,z))]^r,$$

where $\check{\theta}(u) = e^{\sqrt{u}}$. Since all the conditions of Theorem 3 are verified, h has a fixed point. Hence, $(\mathcal{P})$ has a solution in X. $\square$

8. Conclusion. In closing, various applications of fixed point results in the DCMLS were presented throughout this work. Indeed, we proved the existence of solution of two types of differential equations: second order and fourth order. The resolution was based on fixed point theorem, previously proven under suitable assumptions, for each equation. Moreover, we introduce the notion of G_ζ-contraction by combining the graph theory and the notion of fixed point. We explore the uniqueness and the existence of fixed point for such contractions in a DCMLS involving a graph.

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Authors' information:

Nizar Souayah — Dr. Sci. in Mathematics, Professor; https://orcid.org/0000-0001-8245-445X, nsouayah@ksu.edu.sa

Zoran D. Mitrović — Dr. Sci. in Mathematics, Professor; https://orcid.org/0000-0001-9993-9082, zoran.mitrovic@etf.unibl.org

Решение некоторых классов граничных задач с помощью результатов, полученных в частично метрических пространствах с двойным контролем

Н. Суая[1,2], *З. Д. Митрович*[3]

[1] Университет короля Сауда,
Саудовская Аравия, 4545, Эр-Рияд, п/я 145111
[2] Университет Туниса,
Тунис, 1938-1007, Тунис, бул. 9 апреля 1938, 92
[3] Университет Баня-Луки,
Босния и Герцеговина, 78000, Баня-Лука, бул. воеводы Петара Бойовича, 1А

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В работе представлены методы решения граничных задач второго и четвертого порядков. Доказываются несколько новых теорем о неподвижной точке в частично метрических пространствах с двойным контролем. Вводится сжимающее отображение G_ζ, которое действует в том же пространстве, дополнительно снабженное графом, связывающим его элементы, и выводятся результаты о неподвижной точке этого отображения. В качестве приложения полученные результаты используются для доказательства существования решений некоторых классов граничных задач второго и четвертого порядков.

Ключевые слова: дифференциальные уравнения, неподвижная точка, теория графов, частично метрические пространства с двойным контролем.

Контактная информация:

Суая Низар — д-р мат. наук, проф.; https://orcid.org/0000-0001-8245-445X, nsouayah@ksu.edu.sa

Митрович Зоран Д. — д-р мат. наук, проф.; https://orcid.org/0000-0001-9993-9082, zoran.mitrovic@etf.unibl.org