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# Differential game with a "life line" under the Grönwall constraint on controls

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We study the pursuit-evasion and "life line" differential games of one pursuer and one evader, whose controls are subjected to constraints given by Grönwall type inequalities. It is said that an evader has been captured by a pursuer if the state of the pursuer coincides with the state of the evader. One of the main aims of this work is to formulate optimal strategies of players and define guaranteed capture time. Here a strategy of parallel convergence (briefly, П-strategy) for the pursuer is suggested and proved that it is optimal for pursuit. To solve the "life line" problem we will investigate dynamics of the attainability domain of players by Petrosyan method, that is for the attainability domain, conditions of embedding in respect to time are given. This work grows and maintains the works of Isaacs, Petrosyan, Pshenichnyi, Azamov and other researchers.

*Keywords*: differential game, pursuer, evader, Grönwall constraint, strategy, parallel pursuit, attainability domain, "life line" game, the Apollonius sphere.

**1. Introduction.** In the theory of differential games, problems of pursuit-evasion occupy a special place due to a number of specific qualities. In the works [1, 2], this quality was clearly manifested in the construction of the fundamental theory of differential games and in a number of model problems. The book [3] contains specific game problems that were discussed in details and proposed for further study. One of these problems is the so-called "life line" problem that was initially formulated and studied for certain special cases in the book ([3], Problem 9.5.1). For the case when controls of both players are subject to geometric constraints, this game has been rather comprehensively studied by L. A. Petrosyan [4]. In the monograph [4], the notion of strategy of parallel pursuit (briefly, П-strategy) was introduced and used to solve the quality problem for the game with a "life line". This strategy proposed in a simple pursuit game with geometric constraints became the starting point for the development of the pursuit method in games with multiple pursuers (see e.g. [5–21]).

In the theory of differential games, control functions are mainly subjected to geometric, integral or mixed constraints (see [22–26]). However, differential type constraints on controls are also arisen in some applied problems such as ecological, technical problems [27, 28].

In the work [29], the concept of Gr-constraint on controls of players, which in a certain sense, generalizes geometric constraints, is introduced. The present work proposes Grönwall type constraints on controls of players for differential games of pursuit-evasion and for solution of the "life line" game by Petrosyan method. The main purpose of this work is to construct the П-strategy of pursuer, and to find the attainability domain of players, and also to give analytical solution of the "life line" problem in this case.

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**2. Statement of problem.** In the present paper, controls of the pursuer and evader are subjected to the following Grönwall constraints [29, 30]:

$$|u(t)| \leqslant \rho_0 + \rho_1 t + k \int_0^t |u(s)| ds \text{ for almost every } t \geqslant 0, \tag{1}$$

$$|v(t)| \leqslant \sigma_0 + \sigma_1 t + k \int_0^t |v(s)| ds \text{ for almost every } t \geqslant 0, \tag{2}$$

respectively, where $\rho_0$, $\sigma_0$, $\rho_1$, $\sigma_1$, $k$ are given positive numbers.

Note that in (1) and (2) and in further constraints, as the norms of the control vectors $u$ and $v$ in the space $\mathbb{R}^n$, we will consider the usual Euclidean norm, i.e. $|u| = \sqrt{u_1^2 + u_2^2 + ... + u_n^2}$, where $u_1, u_2, ...u_n$ are the coordinates of the vector $u$ in the space $\mathbb{R}^n$, and $|v| = \sqrt{v_1^2 + v_2^2 + ... + v_n^2}$, here $v_1, v_2, ...v_n$ are the coordinates of the vector $v$ in the same space $\mathbb{R}^n$ .

**Remark.** For the cases $\rho_1 = \sigma_1 = 0$, the pursuit-evasion and "life line" problems with Grönwall constraints on controls have been completely studied in the work [29].

Let motion equations of pursuer $P$ and evader $E$ be given by the followings:

$$\dot{x} = u, \quad x(0) = x_0, \tag{3}$$

$$\dot{y} = v, \quad y(0) = y_0, \tag{4}$$

correspondingly, where $x, y, x_0, y_0, u, v \in \mathbb{R}^n$, $n \geqslant 2$, $x_0 \neq y_0$.

**Definition 1.** *A function* $u(\cdot) = (u_1(\cdot), u_2(\cdot), ..., u_n(\cdot))$ *is called an admissible pursuer control in game* (3), (4) *if it satisfies condition* (1). *Similarly, a function* $v(\cdot) = (v_1(\cdot), v_2(\cdot), ..., v_n(\cdot))$ *is called an admissible control of the evader in game* (3), (4) *if it satisfies condition* (2).

The set of all admissible controls of the pursuer and the evader is denoted by the symbols $\mathbf{U}_{\mathrm{Gr}}$ and $\mathbf{V}_{\mathrm{Gr}}$, respectively. Then the pairs $\mathbf{U}_{\mathrm{Gr}}$ and $\mathbf{V}_{\mathrm{Gr}}$ form the motion trajectories

$$x(t) = x_0 + \int_0^t u(s) ds, \quad y(t) = y_0 + \int_0^t v(s) ds$$

of the pursuer and the evader, respectively.

**Definition 2.** *A function* $\mathbf{u} : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *is called a strategy of the pursuer if* $\mathbf{u}(t, v)$ *is a Lebesgue measurable function with respect to* $t$ *for each fixed* $v$ *and is a Borel measurable function with respect to* $v$ *for each fixed* $t$.

**Definition 3.** *It is said that a strategy* $\mathbf{u} = \mathbf{u}(t, v)$ *guarantees capture at time moment* $T(\mathbf{u})$ *if at some time* $t^* \in [0, T(\mathbf{u})]$ *an equality* $x(t^*) = y(t^*)$ *is satisfied for any control* $v(\cdot) \in \mathbf{V}_{\mathrm{Gr}}$ *of the evader, here* $x(t)$ *and* $y(t)$ *are the solutions of the initial value problem*

$$\dot{x} = \mathbf{u}(t, v(t)), \ x(0) = x_0,$$
$$\dot{y} = v(t), \qquad y(0) = y_0,$$

*where* $t \geqslant 0$.

**Definition 4.** *A function* $\mathbf{v} : \mathbb{R}_+ \to \mathbb{R}^n$ *is called a strategy of evader if* $\mathbf{v}(t)$ *is a Lebesgue measurable function with respect to* $t$.

**Definition 5.** *We say that a strategy* $\mathbf{v}(t)$ *is called winning for evader in the* Gr-*game of evasion on* $[0, +\infty)$ *if for any control of pursuer* $u(t) \in \mathbf{U}_{\mathrm{Gr}}$ *the condition* $x(t) \neq y(t)$ *holds for all* $t \geqslant 0$*, here* $x(t)$ *and* $y(t)$ *are the solutions of the initial value problems*

$$\dot{x} = u(t), \ x(0) = x_0,$$

$$\dot{y} = \mathbf{v}(t), \ y(0) = y_0.$$

We use the following statement.

**Lemma** (see [31]). *If* $|\omega(t)| \leqslant \alpha + \int\limits_0^t (\beta + \gamma|\omega(s)|)ds$*, then* $|\omega(t)| \leqslant \frac{\beta}{\gamma}(e^{\gamma t} - 1) + \alpha e^{\gamma t}$*, where* $\omega(t)$*,* $t \geqslant 0$*, is a measurable function, and* $\alpha$*,* $\beta$ *are given non-negative numbers and* $\gamma$ *is a given positive number.*

By this Lemma, if $u(\cdot) \in \mathbf{U}_{\mathrm{Gr}}$ and $v(\cdot) \in \mathbf{V}_{\mathrm{Gr}}$, then

$$|u(t)| \leqslant \varphi(t), \quad t \geqslant 0, \tag{5}$$

$$|v(t)| \leqslant \psi(t), \quad t \geqslant 0, \tag{6}$$

where

$$\varphi(t) = \frac{\rho_1}{k}(e^{kt} - 1) + \rho_0 e^{kt}, \quad \varphi(0) = \rho_0, \tag{7}$$

$$\psi(t) = \frac{\sigma_1}{k}(e^{kt} - 1) + \sigma_0 e^{kt}, \quad \psi(0) = \sigma_0. \tag{8}$$

It can be easily checked that the converse is not true, that is, the inequalities (5) and (6) do not imply (1) and (2). To define the notions of optimal strategies of players and optimal pursuit time, we consider two games.

The goal of the pursuer $P$ is to capture evader $E$, i.e. achievement of the equality $x(t) = y(t)$ (Pursuit problem) and the evader $E$ strives to avoid an encounter (Evasion problem), i.e., to achieve the inequality $x(t) \neq y(t)$ for all $t \geqslant 0$, and in the opposite case, to postpone the instant of encounter as long as possible.

This paper is devoted to solving the following problems for Grönwall type constraints on controls.

**Problem 1.** Solve Pursuit problem in the game (3), (4) with the Grönwall type constraints (1) and (2) (briefly, Gr-game of Pursuit).

**Problem 2.** Solve Evasion problem in the game (3), (4) with the Grönwall type constraints (1) and (2) (briefly, Gr-game of Evasion).

**Problem 3.** Solve the differential game of "life line".

**3. A solution of the Pursuit problem.** In this section, we construct the optimal strategy for pursuer and give a solve of the Pursuit problem.

To construct a strategy for pursuer, first we assume that pursuer knows $t$, $v(t)$ at the current time $t$.

**Definition 6.** *If* $\delta_0 \geqslant 0$*,* $\delta_1 \geqslant 0$*, then the function*

$$\mathbf{u}_{\mathrm{Gr}}(t, v) = v - r(t, v)\xi_0 \tag{9}$$

*is called a* $\Pi_{\mathrm{Gr}}$-*strategy of the pursuer in the* Gr-*game of pursuit, where* $r(t, v) = \langle v, \xi_0 \rangle + \sqrt{\langle v, \xi_0 \rangle^2 + \varphi^2(t) - |v|^2}$*,* $\xi_0 = z_0/|z_0|$*,* $\delta_0 = \rho_0 - \sigma_0$*,* $\delta_1 = \rho_1 - \sigma_1$*,* $z_0 = x_0 - y_0$*,* $\langle v, \xi_0 \rangle$ *is the scalar product of the vectors* $v$ *and* $\xi_0$ *in the Euclidean space* $R^n$*.*

Note that

$$|\mathbf{u}_{\mathrm{Gr}}(t, v)| = \varphi(t). \tag{10}$$

Indeed, if we square equalities (9) on both sides, then we get

$$|\mathbf{u}_{\mathrm{Gr}}(t,v)|^2 = |v|^2 + r(t,v)(r(t,v) - 2\langle v, \xi_0\rangle).$$

From here and from the form of the scalar function $r(t,v)$, it is easy to calculate equality (10). Let us now check the admissibility of strategy (9) for every admissible function $v(t) \in \mathbb{R}^n$, $t \geqslant 0$. From inequality (1) and from equalities (7) and (10) we have

$$\rho_0 + \rho_1 t + k\int\limits_0^t |\mathbf{u}_{\mathrm{Gr}}(v(s),s)|ds = \rho_0 + \rho_1 t + k\int\limits_0^t \varphi(s)ds =$$

$$= \rho_0 + \rho_1 t + k\int\limits_0^t \left[\frac{\rho_1}{k}(e^{ks}-1) + \rho_0 e^{ks}\right]ds = \varphi(t) = |\mathbf{u}_{\mathrm{Gr}}(v(t),t)|,$$

which proves the admissibility of strategy (9).

**Proposition 1.** *If $\delta_0 \geqslant 0$, $\delta_1 \geqslant 0$, then the function $r(t,v)$ is continuous and nonnegative for all $(t,v) \in [0,\infty) \times \mathbb{R}^n$.*

**Proposition 2.** *For every $z_0$, $z_0 \neq 0$, and $v(\cdot) \in \mathbf{V}_{\mathrm{Gr}}$, there exists a scalar function $R(t, v(\cdot))$ such that $z(t) = z_0 R(t, v(\cdot))$, where $z(t) = x(t) - y(t)$.*

**Proposition 3.** *Let $\Phi(t) = A(1-e^{kt}) + Bt + 1$, where $A = \frac{\delta_1 + k\delta_0}{k^2|z_0|}$, $B = \frac{\delta_1}{k|z_0|}$. If $\delta_0 \geqslant 0$, $\delta_1 > 0$ or $\delta_0 > 0$, $\delta_1 \geqslant 0$ is valid, then the function $\Phi(t)$ is monotone decreasing in $t, t \geqslant 0$, and there exists unique positive root of the equation*

$$\Phi(t) = 0 \tag{11}$$

*with respect to $t$. Here we call a guaranteed capture time the positive root of equation* (11) *and denote it by $T_{\mathrm{Gr}}$.*

We prove the statements.

**Theorem 1.** *If $\delta_0 \geqslant 0$, $\delta_1 > 0$ or $\delta_0 > 0$, $\delta_1 \geqslant 0$ is valid, then the $\Pi_{\mathrm{Gr}}$-strategy guarantees capture in the* Gr-*game of pursuit on the time interval* $[0, T_{\mathrm{Gr}}]$.

P r o o f. Let $v(\cdot) \in \mathbf{V}_{\mathrm{Gr}}$ be an arbitrary control of the evader, and let the pursuer use the $\Pi_{\mathrm{Gr}}$-strategy. Using the equations (3) and (4) we get the initial value problem

$$\dot{z} = \mathbf{u}_{\mathrm{Gr}}(t, v(t)) - v(t) = -r(t, v(t))\xi_0, \quad z(0) = z_0.$$

From this, we can see that

$$z(t) = R(t, v(\cdot))z_0, \tag{12}$$

where

$$R(t, v(\cdot)) = 1 - \frac{1}{|z_0|}\int\limits_0^t r(s, v(s))ds.$$

Now we will study the behavior of the function $R(t, v(\cdot))$ of $t$. Using the definition of the function $r(t,v)$, we have

$$R(t, v(\cdot)) \leqslant 1 - \frac{1}{|z_0|}\int\limits_0^t [\sqrt{\langle v(s), \xi_0\rangle^2 + \varphi^2(s) - |v(s)|^2} - |\langle v(s), \xi_0\rangle|]ds.$$

Since the function $f(t,\varsigma)=\sqrt{\varsigma^2+\varphi^2(t)-|v(t)|^2}-\varsigma$, $\varsigma\in\mathbb{R}$, is monotone decreasing with respect to $\varsigma$ for every $t\geqslant 0$. Therefore, by the inequality $|\langle v(t),\xi_0\rangle|\leqslant|v(t)|\leqslant\psi(t)$, and also from (7) and (8) we have

$$R(t,v(\cdot))\leqslant 1-\frac{1}{|z_0|}\int\limits_0^t[\varphi(s)-\psi(s)]ds=1-\frac{1}{|z_0|}\left[\left(\frac{\delta_1}{k^2}+\frac{\delta_0}{k}\right)(e^{kt}-1)-\frac{\delta_1}{k}t\right]=\Phi(t)$$

or

$$R(t,v(\cdot))\leqslant\Phi(t). \tag{13}$$

According to Proposition 3, there is some time $T_{\mathrm{Gr}}$ such that $\Phi(T_{\mathrm{Gr}})=0$. Consequently, from (13) there exists time $t^*$, $0\leqslant t^*\leqslant T_{\mathrm{Gr}}$, that $R(t^*,v(\cdot))=0$, and hence $z(t^*)=0$ by (12).

Next, we will prove the admissibility of the strategy (9) for all $t$, $t\geqslant 0$.

It is easy to check that the equality is valid

$$\dot\varphi(t)=k\varphi(t)+\rho_1.$$

Integrate both sides of this equality

$$\varphi(t)=\rho_0+\rho_1 t+k\int\limits_0^t\varphi(s)ds.$$

Take into account of (10)

$$|\mathbf{u}_{\mathrm{Gr}}(t,v(t)|=\rho_0+\rho_1 t+k\int\limits_0^t|\mathbf{u}_{\mathrm{Gr}}(s,v(s))|ds.$$

This finishes the proof of Theorem 1. □

**Theorem 2.** *If conditions of the Theorem 1 hold, then for any control of the pursuer the strategy of the evader* $\mathbf{v}(t)=-\psi(t)\xi_0$, $t\geqslant 0$, *guarantees to keep the inequality* $x(t)\neq y(t)$ *on the time interval* $[0,T_{\mathrm{Gr}})$.

P r o o f. Let $0\leqslant t<T_{\mathrm{Gr}}$. Then

$$\langle x(t)-y(t),\ \xi_0\rangle=|y_0-x_0|-\int\limits_0^t\langle v(s),\xi_0\rangle ds+\int\limits_0^t\langle u(s),\xi_0\rangle ds\geqslant$$

$$\geqslant|y_0-x_0|+\int\limits_0^t\psi(s)ds-\int\limits_0^t\varphi(s)ds>0.$$

Hence, $x(t)\neq y(t)$, $0\leqslant t<T_{\mathrm{Gr}}$. This completes the proof. □

Theorems 1 and 2 allow us to conclude that $T_{\mathrm{Gr}}$ is the optimal pursuit time, the $\Pi_{\mathrm{Gr}}$-strategy is an optimal strategy for pursuer and $\mathbf{v}(t)=-\psi(t)\xi_0$ is an optimal strategy for the evader $E$.

**4. A solution of the Evasion problem.** In the present section, the Evasion problem is considered as a control problem from the point of view of the evader $E$. To solve this problem we suggest a strategy for the evader $E$ and give a definition of solution of evasion.

**Definition 7.** *We call a strategy of the evader the following control function*:

$$\mathbf{v}_{\mathrm{Gr}}(t) = -\psi(t)\xi_0,\ t \geqslant 0, \tag{14}$$

*in the* Gr-*game of evasion.*

We prove the following statement.

**Theorem 3.** *If* $\delta_0 \leqslant 0,\ \delta_1 \leqslant 0$, *then the strategy* (14) *is winning for the evader in the* Gr-*game of evasion.*

P r o o f. Let $\delta_0 \leqslant 0,\ \delta_1 \leqslant 0$, and $u(\cdot) \in \mathbf{U}_{\mathrm{Gr}}$. Suppose that the evader implements strategy (14) for all $t \geqslant 0$. Obviously, $\mathbf{v}_{\mathrm{Gr}}(t) \in \mathbf{V}_{\mathrm{Gr}}$. Then for any $u(t)$ we obtain

$$|z(t)| \geqslant \left| z_0 - \int\limits_0^t \mathbf{v}_{\mathrm{Gr}}(s)ds \right| - \int\limits_0^t |u(s)|ds = |z_0| + \int\limits_0^t \psi(s)ds - \int\limits_0^t |u(s)|ds.$$

Using the inequality $|u(t)| \leqslant \varphi(t)$ obtained

$$|z(t)| \geqslant \Psi(t),$$

where $\Psi(t) = |z_0| + ((\delta_1 + k\delta_0)/k^2)(1 - e^{kt}) + (\delta_1/k)t$.

If $\delta_0 \leqslant 0,\ \delta_1 \leqslant 0$, and $k > 0$, then

$$\frac{d\Psi(t)}{dt} = \frac{\delta_1}{k} - \left(\frac{\delta_1}{k} + \delta_0\right) e^{kt} \geqslant 0$$

for all $t \geqslant 0$.

This implies that the function $\Psi(t)$ is monotone increasing on $[0, \infty)$. Hence it follows that $\Psi(t) \geqslant |z_0| > 0$. This completes the proof of Theorem 3. □

**5. The differential game with "life line".** Here we are going to study mainly the game with phase constraints for the evader being given by a subset $M$ of $\mathbb{R}^n$ which is called the "life line" (for the evader naturally). (Notice that in the case $M = \emptyset$ we have a simple game.)

In the the Differential Game with "life line" the pursuer $P$ aims to catch the evader $E$, i.e. to realize the equality $x(t) = y(t)$ for some $t > 0$, while the evader $E$ stays in the zone $\mathbb{R}^n \setminus M$. The aim of the evader $E$ is to reach the zone $M$ before being caught by the pursuer $P$ or to keep the relation $x(t) \neq y(t)$ for all $t,\ t \geqslant 0$. Notice that $M$ doesn't restrict motion of the pursuer $P$. Further we will assume initial positions $x_0$ and $y_0$ are given such that $x_0 \neq y_0$ and $y_0 \notin M$.

**Definition 8.** *A strategy* $\mathbf{u}_{\mathrm{Gr}}(t, v)$ *of the pursuer* $P$ *is called winning on the time interval* $[0, T_{\mathrm{Gr}}]$ *in the game of "life line" if for every* $v(\cdot) \in \mathbf{V}_{\mathrm{Gr}}$ *there exists some moment* $t^* \in [0, T_{\mathrm{Gr}}]$ *that* $x(t^*) = y(t^*)$ *and* $y(t) \notin M$ *while* $t \in [0, t^*]$.

**Definition 9.** *A control function* $v^*(\cdot) \in \mathbf{V}_{\mathrm{Gr}}$ *of the evader* $E$ *is called winning in the game of "life line" if for every* $u(\cdot) \in \mathbf{U}_{\mathrm{Gr}}$ : *there exists some moment* $\overline{t},\ \overline{t} > 0$, *such that* $y(\overline{t}) \in M$ *and* $x(t) \neq y(t)$ *while* $t \in [0, \overline{t})$, *or* $x(t) \neq y(t)$ *for all* $t \geqslant 0$.

***5.1. Dynamics of the attainability domain.*** Suppose that $\delta_0 > 0,\ \delta_1 \geqslant 0$. In consequence, a set of capture points may consist of some finite set. We will construct the attainability domain under these conditions.

Assume that the evader $E$ chooses any control function $v(\cdot) \in \mathbf{V}_{\mathrm{Gr}}$ and the pursuer $P$ applies the strategy (9). Define for each control the following trajectories of the evader

$E$ and pursuer $P$:

$$y(t) = y_0 + \int\limits_0^t v(s)ds, \quad x(t) = x_0 + \int\limits_0^t \mathbf{u}_{\mathrm{Gr}}(s, v(s))ds$$

on interval $t \in [0, \tau]$ respectively, where $\tau$ is a pursuit time.

Now we generate the sets

$$B_P(t) = B_P(x(t), y(t)) = \{p : |p - x(t)| \geqslant h(t)|p - y(t)|\}, \tag{15}$$

$$B_P(0) = B_P(x_0, y_0) = \{p : |p - x_0| \geqslant h(0)|p - y_0|\} \tag{16}$$

for the pair $(x(t), y(t))$, where

$$h(t) = \frac{\varphi(t)}{\psi(t)} = \frac{\frac{\rho_1}{k}(e^{kt} - 1) + \rho_0 e^{kt}}{\frac{\sigma_1}{k}(e^{kt} - 1) + \sigma_0 e^{kt}}, \quad h(0) = \frac{\rho_0}{\sigma_0}.$$

Here it is obvious that the relation $y(t) \in B_P(t)$ holds for each $t \in [0, \tau]$.

**Proposition 4.** *If $\delta_0 > 0$, $\delta_1 \geqslant 0$, then for the scalar function $h(t)$ the relation $h(t) > 1$ holds on the time interval $[0, \tau]$.*

**Theorem 4.** *The set* (15) *is equivalent to*

$$B_P(t) = x(t) + R(t, v(\cdot))[a(t, z_0)S + c(t, z_0)] \tag{17}$$

*for all $t \in [0, \tau]$, here $S$ is the unit ball whose center is on zero point in $\mathbb{R}^n$ and*

$$a(t, z_0) = \frac{h(t)|z_0|}{h^2(t) - 1}, \quad c(t, z_0) = -\frac{h^2(t) z_0}{h^2(t) - 1}.$$

P r o o f. From (15) we get

$$B_P(t) = x(t) + B_P^*(t), \tag{18}$$

here $B_P^*(t) = \{p : |p| \geqslant h(t)|p + z(t)|\}$. Now we will present that $B_P^*(t)$ is a ball. For this purpose, square both sides of the inequality

$$|p| \geqslant h(t)|p + z(t)|,$$

and after that simplify the last result, i.e.

$$|p|^2 \geqslant h^2(t)\left(|p|^2 + 2\langle p, z(t)\rangle + |z(t)|^2\right)$$

or

$$(h^2(t) - 1)|p|^2 + 2h^2(t)\langle p, z(t)\rangle + h^2(t)|z(t)|^2 \leqslant 0. \tag{19}$$

Divide both sides of (19) by the expression $h^2(t) - 1$

$$|p|^2 + \frac{2h^2(t)\langle p, z(t)\rangle}{h^2(t) - 1} + \frac{h^2(t)|z(t)|^2}{h^2(t) - 1} \leqslant 0. \tag{20}$$

Add the expression $\left(\frac{h^2(t)z(t)}{h^2(t)-1}\right)^2$ to both sides of (20), and write it down as

$$|p|^2 + 2\left\langle p,\ \frac{h^2(t)z(t)}{h^2(t)-1}\right\rangle + \left(\frac{h^2(t)z(t)}{h^2(t)-1}\right)^2 \leqslant \left(\frac{h^2(t)z(t)}{h^2(t)-1}\right)^2 - \frac{h^2(t)|z(t)|^2}{h^2(t)-1}.$$

After some simplification we can generate the result

$$\left|p + \frac{h^2(t)z(t)}{h^2(t)-1}\right| \leqslant \frac{h(t)|z(t)|}{h^2(t)-1}.$$

Hence we have set

$$B_P^*(t) = \{p : |p - c(t, z(t))| \leqslant a(t, z(t))\} = c(t, z(t)) + a(t, z(t))S,$$

where $B_P^*(t)$ is the ball whose center is on the point $c(t, z(t)) = -\frac{h^2(t)z(t)}{h^2(t)-1}$ and whose radius equals $a(t, z(t)) = \frac{h(t)|z(t)|}{h^2(t)-1}$. Then from formula (12) we obtain

$$c(t, z(t)) = -\frac{h^2(t)z(t)}{h^2(t)-1} = -R(t, v(\cdot))\frac{h^2(t)z_0}{h^2(t)-1},$$

$$a(t, z(t)) = \frac{h(t)|z(t)|}{h^2(t)-1} = R(t, v(\cdot))\frac{h(t)|z_0|}{h^2(t)-1}.$$

In consequence, the (18) can be written in the form

$$B_P(t) = x(t) + R(t, v(\cdot))\left[\frac{h(t)|z_0|}{h^2(t)-1}S - \frac{h^2(t)z_0}{h^2(t)-1}\right]$$

or in the form (17) which finishes the proof. □

Now we are going to show monotony of the set $B_P(t)$.

**Theorem 5 (Petrosyan type theorem** [10]**).** *Let: a)* $\rho_0 > \sigma_0,\ \rho_1 > \sigma_1$, *and b)* $\rho_1\sigma_0 \geqslant \rho_0\sigma_1$. *Then the set* $B_P(t)$ *is monotone in relation to the inclusion while* $t \in [0, \tau]$, *i.e.* $B_P(t_1) \supset B_P(t_2)$ *for* $0 \leqslant t_1 \leqslant t_2$.

P r o o f. First, by (17) we determine the derivative of the support function (see [32]) $F(B_P(t), \mu)$ of the set $B_P(t)$ for any $\mu \in \mathbb{R}^n$ and $|\mu| = 1$, that is,

$$\frac{d}{dt}F(B_P(t), \mu) = \frac{d}{dt}F\left(x(t) + R(t, v(\cdot))[a(t, z_0)S + c(t, z_0)], \mu\right) =$$

$$= \frac{d}{dt}\left[\langle x(t), \mu\rangle + R(t, v(\cdot))[a(t, z_0)F(S, \mu) + \langle c(t, z_0), \mu\rangle]\right] =$$

$$= \langle \dot{x}(t), \mu\rangle + \dot{R}(t, v(\cdot))[a(t, z_0) + \langle c(t, z_0), \mu\rangle] +$$

$$+ R(t, v(\cdot))[\dot{a}(t, z_0) + \langle \dot{c}(t, z_0), \mu\rangle] = \Phi_1(t, \mu) + \Phi_2(t, \mu),$$

where

$$\Phi_1(t, \mu) = \langle \dot{x}(t), \mu\rangle + \dot{R}(t, v(\cdot))[a(t, z_0) + \langle c(t, z_0), \mu\rangle],$$

$$\Phi_2(t, \mu) = R(t, v(\cdot))[\dot{a}(t, z_0) + \langle \dot{c}(t, z_0), \mu\rangle].$$

Now we prove that the inequality

$$\frac{d}{dt}F(B_P(t), \mu) = \Phi_1(t, \mu) + \Phi_2(t, \mu) \leqslant 0$$

is true on $t \in (0, \tau]$.

To do this, we first show that $\Phi_1(t,\mu) \leqslant 0$. Square the inequality $|v(t)| \leqslant \psi(t)$ and multiply both sides of the result by the expression $\frac{h^2(t)}{h^2(t)-1}$

$$\frac{|v(t)|^2 h^2(t)}{h^2(t)-1} \leqslant \frac{\varphi^2(t)}{h^2(t)-1}.$$

Make some simplification

$$|v(t)|^2 \left(1+\frac{1}{h^2(t)-1}\right) \leqslant \frac{\varphi^2(t)}{h^2(t)-1} \Rightarrow$$

$$\Rightarrow |v(t)|^2 \leqslant \frac{(\varphi^2(t)-|v(t)|^2)}{h^2(t)-1}. \tag{21}$$

According to definition of the function $r(t,v(t))$ (see Definition 6) we can express the equality $\varphi^2(t)-|v(t)|^2 = r(t,v(t))[r(t,v(t))-2\langle v(t),\xi_0\rangle]$, and make some substitutions in (21)

$$|v(t)|^2 \leqslant \frac{r(t,v(t))}{h^2(t)-1}[r(t,v(t))-2\langle v(t),\xi_0\rangle]$$

or

$$|v(t)|^2 + 2r(t,v(t))\frac{\langle v(t),\xi_0\rangle}{h^2(t)-1} \leqslant \frac{r^2(t,v(t))}{h^2(t)-1}. \tag{22}$$

Add the expression $\frac{r^2(t,v(t))}{(h^2(t)-1)^2}$ to both sides of (22), and rewrite it again

$$|v(t)|^2 + 2r(t,v(t))\frac{1}{h^2(t)-1}\langle v(t),\xi_0\rangle + \frac{r^2(t,v(t))}{(h^2(t)-1)^2} \leqslant$$

$$\leqslant \frac{r^2(t,v(t))}{(h^2(t)-1)^2} + \frac{r^2(t,v(t))}{h^2(t)-1}. \tag{23}$$

The fact that the left-hand side of (23) consists of quadratic standard form of the sum of two vectors, and as a consequence of simplifying the right-hand side of (23) we obtain

$$\left|v(t)+\frac{r(t,v(t))}{h^2(t)-1}\xi_0\right| \leqslant \frac{h(t)r(t,v(t))}{h^2(t)-1}. \tag{24}$$

It is obvious that for any vector $\mu \in \mathbb{R}^n$, $|\mu|=1$ the inequality

$$\left\langle v(t)+\frac{r(t,v(t))}{h^2(t)-1}\xi_0,\ \mu\right\rangle \leqslant \left|v(t)+\frac{r(t,v(t))}{h^2(t)-1}\xi_0\right|$$

is valid.

According to this and from (24), we have

$$\left\langle v(t)+\frac{r(t,v(t))}{h^2(t)-1}\xi_0,\ \mu\right\rangle \leqslant \frac{h(t)r(t,v(t))}{h^2(t)-1} \Rightarrow$$

$$\Rightarrow \langle v(t),\mu\rangle - r(t,v(t))\left(1-\frac{h^2(t)}{h^2(t)-1}\right)\langle \xi_0,\mu\rangle \leqslant \frac{h(t)r(t,v(t))}{h^2(t)-1} \Rightarrow$$

$$\Rightarrow \langle v(t)-r(t,v(t))\xi_0,\ \mu\rangle + \frac{h^2(t)r(t,v(t))}{h^2(t)-1}\langle \xi_0,\mu\rangle \leqslant \frac{h(t)r(t,v(t))}{h^2(t)-1} \Rightarrow$$

$$\Rightarrow \langle \dot{x}(t), \mu\rangle - \frac{r(t, v(t))}{|z_0|}\left(-\frac{h^2(t)}{h^2(t)-1}\langle z_0, \mu\rangle + \frac{h(t)}{h^2(t)-1}|z_0|\right) \leqslant 0 \Rightarrow$$

$$\Rightarrow \langle \dot{x}(t), \mu\rangle + \dot{R}(t, v(\cdot))[a(t, z_0) + \langle c(t, z_0), \mu\rangle] \leqslant 0. \tag{25}$$

Formula (25) means that $\Phi_1(t, \mu) \leqslant 0$.

Now we are going to present that the inequality $\Phi_2(t, \mu) \leqslant 0$ holds. Because of $R(t, v(\cdot)) \geqslant 0$ on $t \in (0, \tau]$, it is enough to prove that

$$\dot{a}(t, z_0) + \langle \dot{c}(t, z_0), \mu\rangle = \left(\frac{h(t)}{h^2(t)-1}\right)'|z_0| - \left\langle \left(\frac{h^2(t)}{h^2(t)-1}\right)' z_0,\ \mu\right\rangle \leqslant 0. \tag{26}$$

First, compute the first derivative on the right-hand side of (26)

$$\left(\frac{h^2(t)}{h^2(t)-1}\right)' = \frac{-2h(t)h'(t)}{(h^2(t)-1)^2}.$$

On the other hand, using (7), (8) and based on the condition b) of Theorem 5 we have the following:

$$\frac{dh(t)}{dt} = \frac{\varphi'(t)\psi(t) - \varphi(t)\psi'(t)}{\psi^2(t)} = \frac{(\rho_1\sigma_0 - \rho_0\sigma_1)e^{kt}}{\psi^2(t)} \geqslant 0. \tag{27}$$

Consequently, this inequality $\left(\frac{h^2(t)}{h^2(t)-1}\right)' \leqslant 0$ is satisfied in the interval $(0, \tau]$.

Now multiply both sides of the inequality $\langle \xi_0, \mu\rangle \leqslant 1$ (for any $\mu \in R^n,\ |\mu| = 1$) by the expression $-\left(\frac{h^2(t)}{h^2(t)-1}\right)'$

$$-\left(\frac{h^2(t)}{h^2(t)-1}\right)'\langle \xi_0, \mu\rangle \leqslant -\left(\frac{h^2(t)}{h^2(t)-1}\right)'. \tag{28}$$

Add $\left(\frac{h(t)}{h^2(t)-1}\right)'$ to both sides of (28), and obtain

$$\left(\frac{h(t)}{h^2(t)-1}\right)' - \left(\frac{h^2(t)}{h^2(t)-1}\right)'\langle \xi_0, \mu\rangle \leqslant \left(\frac{h(t)}{h^2(t)-1}\right)' - \left(\frac{h^2(t)}{h^2(t)-1}\right)'. \tag{29}$$

Compute the right-hand side of (29) and in relation to (27) we obtain this result

$$\left(\frac{h(t)}{h^2(t)-1}\right)' - \left(\frac{h^2(t)}{h^2(t)-1}\right)' = -\frac{h'(t)}{(h(t)+1)^2} \leqslant 0.$$

From this, we generate the following relation for the left-hand side of (29):

$$\left(\frac{h(t)}{h^2(t)-1}\right)' - \left(\frac{h^2(t)}{h^2(t)-1}\right)'\langle \xi_0, \mu\rangle \leqslant 0. \tag{30}$$

Multiply both sides of (30) by $|z_0|$ and we get (26). Hence it follows that $\Phi_2(t, \mu) \leqslant 0$. This completes the proof. □

***5.2. A solution of the game with "life line".*** It has been noted above that Isaacs' game with "life line" was comprehensively solved by L. A. Petrosyan using the method of approximating measurable controls by piecewise constant.

**Theorem 6.** *If Theorem 5 is valid and* $M \bigcap B_P(0) = \emptyset$, *then the* $\Pi_{\rm Gr}$*-strategy* (9) *is winning in the game* (1)–(4) *with “life line”.*

P r o o f. The proof directly follows from Theorem 5. $\square$

Now we define a set in the form

$$B_E(t) = \{p: \ |p - x_0| \geqslant \chi(t)|p - y_0|\}$$

for all $t \in (0, T_{\rm Gr}]$, where

$$\chi(t) = \frac{\int\limits_0^t \varphi(s)ds}{\int\limits_0^t \psi(s)ds} = \frac{(\rho_1 + k\rho_0)(e^{kt} - 1) - k\rho_1 t}{(\sigma_1 + k\sigma_0)(e^{kt} - 1) - k\sigma_1 t}$$

and $T_{\rm Gr}$ is the first positive root of the equation (11). It's obvious that $\lim_{t\to 0} \chi(t) = \frac{\rho_0}{\sigma_0}$.

**Theorem 7.** *Let the conditions* $\rho_0 > \sigma_0$, $\rho_1 > \sigma_1$ *and* $\rho_1\sigma_0 \geqslant \rho_0\sigma_1$. *Then* $\chi(t)$ *is increasing on interval* $t \in (0, T_{\rm Gr}]$ *and the set* $B_E(t)$ *is decreasing with respect to* $t \in (0, T_{\rm Gr}]$, *i.e., an inclusion* $B_E(t_1) \supset B_E(t_2)$ *holds for any* $t_1, t_2 \in (0, T_{\rm Gr}]$ *and* $0 < t_1 \leqslant t_2$.

P r o o f. First of all, we will prove that $\chi(t)$ is increasing, i.e. $\chi'(t) \geqslant 0$ under the conditions of Theorem 7. For this purpose, calculate the derivative of $\chi(t)$

$$\frac{d\chi(t)}{dt} = \frac{k^2(\rho_0\sigma_1 - \rho_1\sigma_0)(e^{kt} - 1 - ke^{kt}t)}{[(\sigma_1 + k\sigma_0)(e^{kt} - 1) - k\sigma_1 t]^2}.$$

Now, analyze a sign of expression $l(t) = e^{kt} - 1 - ke^{kt}t$ for every $t \in (0, T_{\rm Gr}]$, i. e.: a) $\lim_{t\to 0} l(t) = 0$; b) $\frac{dl(t)}{dt} = -k^2 e^{kt} t \leqslant 0$. So, $l(t) \leqslant 0$ on interval $t \in (0, T_{\rm Gr}]$.

Therefore, $\chi^{'}(t) \geqslant 0$ is true if $\rho_1\sigma_0 \geqslant \rho_0\sigma_1$. Since $\chi(t)$ is increasing on $t \in (0, T_{\rm Gr}]$, we can write a relation $\chi(t) > 1$ on that interval.

From (15), (16), we have

$$B_E(t) = x_0 - \frac{\chi^2(t)}{\chi^2(t) - 1} z_0 + \frac{\chi(t)}{\chi^2(t) - 1} |z_0| S. \quad (31)$$

Then we determine the character of the derivative of support function $F(B_E(t), \mu)$, when $|\mu| = 1$ and $t \in (0, T_{\rm Gr}]$:

$$\frac{d}{dt} F(B_E(t), \mu) = -\left(\frac{\chi^2(t)}{\chi^2(t) - 1}\right)' \langle z_0, \mu\rangle + \left(\frac{\chi(t)}{\chi^2(t) - 1}\right)' |z_0| =$$

$$= \frac{2\chi(t)}{(\chi^2(t) - 1)^2} \chi'(t) \langle z_0, \mu\rangle - \frac{\chi^2(t) + 1}{(\chi^2(t) - 1)^2} \chi'(t) |z_0| =$$

$$= (2\chi(t)\langle \xi_0, \mu\rangle - \chi^2(t) - 1) \frac{\chi'(t)|z_0|}{(\chi^2(t) - 1)^2} =$$

$$= -|\xi_0 \chi(t) - \mu|^2 \frac{\chi'(t)|z_0|}{(\chi^2(t) - 1)^2} \leqslant 0.$$

$\square$

**Theorem 8.** *Let* $\delta_0 > 0$, $\delta_1 > 0$, $\rho_1\sigma_0 \geqslant \rho_0\sigma_1$ *and* $M \bigcap B_E(T_{\rm Gr}) \neq \emptyset$. *Then for evader* $E$ *there exists some control which is winning in the game* (1)–(4) *with “life line”.*

P r o o f. Let $p \in M \bigcap B_E(T_{\rm Gr})$ and the evader $E$ implements the control $v^*(t) = \psi(t)\nu,\ v^*(\cdot) \in \mathbf{V}_{\rm Gr}$, where $\nu = (p - y_0)/|p - y_0|$. Since $|v^*(t)| = \psi(t)$ for all $t \geqslant 0$, then substituting this into inequality (2) we get formula

$$\sigma_0 + \sigma_1 t + k\int\limits_0^t |v^*(s)|ds = \sigma_0 + \sigma_1 t + k\int\limits_0^t \psi(s)ds =$$

$$= \sigma_0 + \sigma_1 t + k\int\limits_0^t \left[\frac{\sigma_1}{k}(e^{ks} - 1) + \sigma_0 e^{ks}\right] ds = \psi(t) = |v^*(t)|,$$

which proves the admissibility of the control $v^*(t)$ for all $t \geqslant 0$. Then time of the achievement of the point $p$ is $\bar\eta$ for evader and we have

$$\int\limits_0^{\bar\eta} |v^*(s)|ds = \int\limits_0^{\bar\eta} \psi(s)ds = |p - y_0|, \tag{32}$$

and from of Theorems 1 and 7, it follows, that $\bar\eta \leqslant T_{\rm Gr}$. We suppose that for the pursuer exists a certain control function $u^*(\cdot) \in \mathbf{U}_{\rm Gr}$ that $x(\bar t) = y(\bar t)$ holds and $\bar t < \bar\eta$. If $z(t) = x(t) - y(t)$ and $z(0) = z_0$, then from $\dot z(t) = u^*(t) - v^*(t)$ we have

$$z(\bar t) = z_0 + \int\limits_0^{\bar t} (u^*(s) - v^*(s))ds = 0.$$

From this, it follows that

$$\left| z_0 - \int\limits_0^{\bar t} v^*(s)ds \right| \leqslant \int\limits_0^{\bar t} |u^*(s)|ds \leqslant \int\limits_0^{\bar t} \varphi(s)ds \Rightarrow$$

$$\Rightarrow |z_0|^2 - 2\int\limits_0^{\bar t} \psi(s)ds\langle z_0, \nu\rangle + \left(\int\limits_0^{\bar t} \psi(s)ds\right)^2 \leqslant \left(\int\limits_0^{\bar t} \varphi(s)ds\right)^2 \Rightarrow$$

$$\Rightarrow \left(\int\limits_0^{\bar t} \psi(s)ds\right)^2 (\chi^2(\bar t) - 1) + 2\int\limits_0^{\bar t} \psi(s)ds\langle z_0, \nu\rangle - |z_0|^2 \geqslant 0 \Rightarrow$$

$$\Rightarrow \int\limits_0^{\bar t} \psi(s)ds \geqslant f(\bar t) := \frac{1}{\chi^2(\bar t) - 1}[\sqrt{\langle z_0, \nu\rangle^2 + |z_0|^2(\chi^2(\bar t) - 1)} - \langle z_0, \nu\rangle]. \tag{33}$$

If $\chi(t)$ is increasing for $t \in (0, T_{\rm Gr}]$, then it is easy to check that $f(t)$ is decreasing function on $(0, T_{\rm Gr}]$. Consequently from $\bar t < \bar\eta$, it follows that $f(\bar\eta) \leqslant f(\bar t)$.

Since $p \in B_E(T_{\rm Gr})$ and $\bar\eta \leqslant T_{\rm Gr}$, then from Theorem 7 we have $p \in B_E(T_{\rm Gr}) \subset B_E(\bar\eta)$. Hence we obtain

$$|p - x_0| \geqslant \chi(\bar\eta)|p - y_0| \Rightarrow$$

$$\Rightarrow |z_0 - (p - y_0)|^2 \geqslant \chi^2(\bar\eta)|p - y_0|^2 \Rightarrow$$

$$\Rightarrow |z_0|^2 - 2\langle z_0, p - y_0\rangle + |p - y_0|^2 \geqslant \chi^2(\bar\eta)|p - y_0|^2 \Rightarrow$$
$$\Rightarrow 0 \geqslant (\chi^2(\bar\eta) - 1)|p - y_0|^2 + 2|p - y_0|\langle z_0, \nu\rangle - |z_0|^2 \Rightarrow$$
$$\Rightarrow |p - y_0| \leqslant f(\bar\eta) \leqslant f(\bar t).$$

Then from the last inequality and from (32), (33), we have $\int_0^{\bar t} \psi(s)ds \geqslant \int_0^{\bar\eta} \psi(s)ds$ or $\bar t \geqslant \bar\eta$, though this contradicts our supposition. □

**Theorem 9.** *Let* $\delta_0 \leqslant 0,\ \delta_1 \leqslant 0$. *Then for the evader* $E$ *there exists some control which is winning in the game* (1)–(4) *with "life line".*

P r o o f. Let the evader use the control (14), and let the pursuer choose an arbitrary control $u(\cdot) \in \mathbf{U}_{\mathrm{Gr}}$. Then, similar to the proof of Theorem 3, under the conditions of the current theorem we again derive $|z(t)| \geqslant \Psi(t) \geqslant |z_0| > 0$ for any $t \in [0, +\infty)$, from which we infer $z(t) \neq 0$, i.e. $x(t) \neq y(t)$ (see Proposition 2). Therefore, by virtue of Definition 9 in the game (1)–(4) with a "life line" the evader $E$ is also considered to be winning. The proof is complete. □

**6. Example.** Assume that the game (1)–(4) is described as (see Figures 1 and 2)

$$\dot x = u, \quad x_0 = (0, 0), \quad |u(t)| \leqslant 2 + 2\sqrt{2}t + \int_0^t |u(s)|ds, \quad t \geqslant 0, \tag{34}$$

$$\dot y = v, \quad y_0 = (0, -1), \quad |v(t)| \leqslant \sqrt{3} + 2t + \int_0^t |v(s)|ds, \quad t \geqslant 0. \tag{35}$$

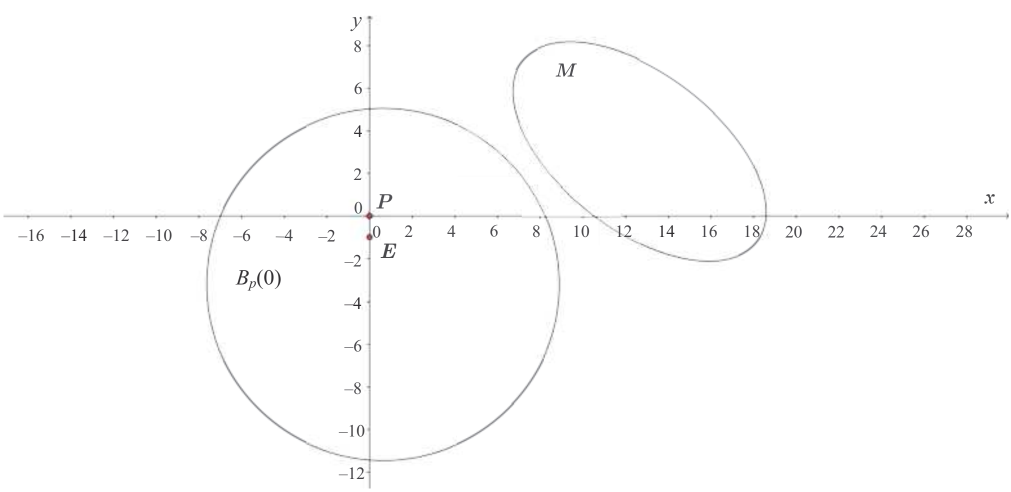


*Figure 1.* The representation which the $\Pi_{\mathrm{Gr}}$-strategy is winning in the game (34), (35) with "life line"

Then based on Theorem 1, we can obtain $T_{\mathrm{Gr}} = 0.37$. In accordance with Theorem 4, we generate the set $B_P(0) = \{p : |p - c| \leqslant a,\ c = (0, -4),\ a = 4 + 2\sqrt{3}\}$. A set of points $p = (\tilde p_1, \tilde p_2)$ on a boundary of $B_P(0)$ consists of the circle

$$\partial B_P(0) = \{(\tilde p_1, \tilde p_2) : \tilde p_1^2 + (\tilde p_2 + 4)^2 = 28 + 16\sqrt{3}\}.$$

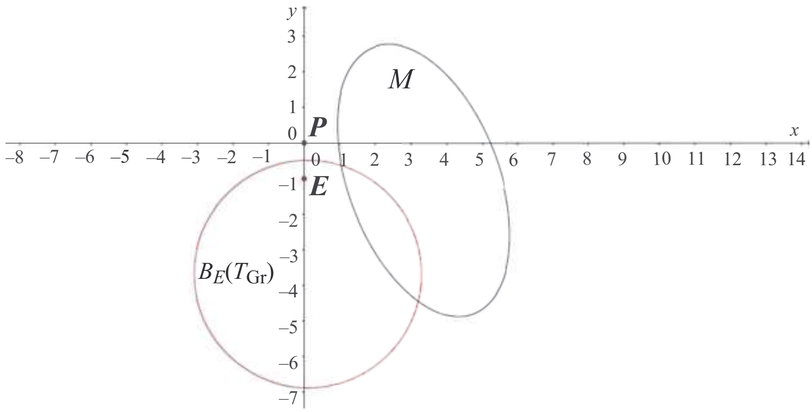


*Figure 2.* The representation which the evader wins
in the game (34), (35) with "life line"

Using (31) and Theorem 7, the set $B_E(T_{\text{Gr}}) = \{p : |p - c| \leqslant a,\ c = (0, -3.323),\ a = 2.779\}$ is constructed. A set of points $p = (\hat{p}_1, \hat{p}_2)$ on a boundary of $B_E(T_{\text{Gr}})$ consists of the following circle:

$$\partial B_E(T_{\text{Gr}}) = \{(\hat{p}_1, \hat{p}_2) : \hat{p}_1^2 + (\hat{p}_2 + 3.323)^2 = (2.779)^2\}.$$

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## Дифференциальная игра с «линией жизни» при ограничениях Гронуолла на управления

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Изучается дифференциальная игра с «линией жизни» для одного преследователя и одного убегающего при управлениях удовлетворяющих неравенств типа Грануолла. Убегающий считается пойманным со стороны преследователя, если состояние убегающего совпадает с состоянием преследователя. Одна из основных целей настоящей работы — построение оптимальных стратегий для игроков и определение оптимального времени поимки. Для преследователя предлагается стратегия параллельного сближения (короче, П-стратегия) и доказывается ее оптимальность. Для решения задачи с «линией жизни» исследуется динамика области достижимости игроков методом Петросяна, т. е. найдены условия монотонности по включению относительно времени этой области достижимости. Отметим, что работа продолжает исследования Айзекса, Петросяна, Пшеничного, Азамова и др.

*Ключевые слова*: дифференциальная игра, преследование, убегание, ограничение Гроноулла, стратегия, параллельное преследование, область достижимости, игра с «линией жизни», сфера Аполлония.

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