UDC 519.83
MSC 91A25
Вестник СПбГУ. Прикладная математика. Информатика... 2024. Т. 20. Вып. 1

A note on cooperative differential games with pairwise interactions*

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For citation: He Y., Petrosyan L. A. A note on cooperative differential games with pairwise interactions. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2024, vol. 20, iss. 1, pp. 91–108.
https://doi.org/10.21638/11701/spbu10.2024.108

In this paper, a differential game with pairwise interaction in a network is proposed. For explicitly, the vertices are players, and the edges are connections between them. Meanwhile, we consider the cooperative case. One special characteristic function is introduced and its convexity is proved. The core is used as a cooperative optimality principle. The characteristic function allows the construction of a time-consistent (dynamically stable) solutions, such as the Shapley value and the core. Finally, the results are illustrated by an example.

Keywords: cooperative games, differential network games, pairwise interactions, characteristic function, the Shapley value, time-consistency.

1. Introduction. The pairwise interaction games are new and important part of modern game theory. The first study on pairwise interaction in non-cooperative cases in network structure is done in [1]. In other words, the pairwise interaction games are proper subclass of the usual graphical or network game [2]. In the case of two strategies, pairwise interaction games on the complete graph can be modeled as congestion games [3].

To the best of the author's knowledge, the research related to cooperative games with pairwise interaction is done in [4–8]. For the first time, in a cooperative form, a multistage network game with pairwise interaction is considered, when players play bimatrix games with their neighbors by network structure, and sufficient conditions for strong time consistency of the core are formulated [4]. Also, in [5], for a particular class of symmetric networks (star-network), a simplified formula for calculating the components of the Shapley value is obtained, and conditions for strong time consistency of the core are derived. In [6], considering multistage cooperative games with pairwise interaction, an analogue of the core is constructed and its strong time consistency proved. Furthermore, alternative approaches to constructing the characteristic function for games with pairwise interaction are considered [7]. And a new characteristic function is constructed, which has a lower computational complexity than the classical one, the IDP-core is proposed and its strong time consistency proved [8].

Since that most real-life game situations are dynamic rather than static, network differential games have become a field that attracts theoretical and technical developments [9–11]. We developed differential game models for studying congested traffic networks [12]. Cooperative differential games on networks are developed in [13]. A time-consistent Shapley value and τ value solution in a class of differential network games was proposed in [13]. Later, Tur and Petrosyan give a simplified formula for the calculation of the Shapley value [14]. Additionally, they also proved that the core is strongly time-consistent [15]. In

* This research was supported by the Russian Science Foundation (project N 22-11-00051, https://rscf.ru/project/22-11-00051/).

cooperative differential game theory, it is important for the solution to be dynamically stable (time-consistent), meaning that players have no intention to break the rules. The notion of time consistency of differential game solutions was first introduced in [16, 17]. The cooperative game model is based on the characteristic function, and in [18], a new the characteristic function was introduced for cooperative differential games on networks.

In the paper, we consider a new class of games which is a subclass of differential network games namely differential games with pairwise interactions. For this class of games, we prove the convexity of the introduced characteristic function which is not true in general and guarantees that the Shapley value belongs to the core. In addition, the Shapley value is time-consistent which is an exception in differential and dynamic cooperative game theory. The theory is illustrated on nontrivial three person differential pollution control game with pairwise interactions. The results of numerical simulation are also presented in this case.

The rest of the paper is organized as follows. In Section 2 a class of differential network games with pairwise interaction is described. In Section 3 a special characteristic functions is introduced. In Section 4 the core and the Shapley value for the considered class of games are introduced. In Section 5 one illustrative example, with different differential games on the various links of the network is considered. In Section 6 the conclusions are drawn.

2. A class of differential network games with pairwise interaction. Consider a class of n-person differential network games with pairwise interaction over the time horizon $[t_0, T]$. The players are connected to a network system. Let $N = \{1, 2, ..., n\}$ denote the set of players in the network. The nodes of the network are used to represent the players in the network.

A pair (N, L) is called a network, where N is a set of nodes, and $L \subset N \times N$ is a given set of arcs. Note that the pair $\text{arc}(i, i) \notin L$. If pair $\text{arc}(i, j) \in L$, denote link as $i \Leftrightarrow j$ connects players i and j, $j \in \widetilde{K}(i)$. It is supposed that all connections are undirected. We also denote the set of players connected to player i as $\widetilde{K}(i) = [j : \text{arc}(i, j) \in L]$, for $i \in N$, $i \neq j$, $K(i) = \widetilde{K}(i) \cup i$.

The state dynamics of the game are given by

$$\dot{x}^{ij}(\tau) = f^{ij}(x^{ij}(\tau), u^{ij}(\tau), u^{ji}(\tau)),\ x^{ij}(t_0) = x_0^{ij} \tag{1}$$

for $\tau \in [t_0; T]$ and $i \in N$, $j \in \widetilde{K}(i)$.

Here $x^{ij}(\tau) \in R^m$ is the state variable of player i interacting with player $j \in \widetilde{K}(i)$ at time τ, and $u^{ij}(\tau) \in U_{ij}$, $U_{ij} \subset \text{Comp}R^l$, the control variable of player i interacting with player j. Every player i plays a differential game with player j according to the network structure. The function $f^{ij}(x^{ij}(\tau), u^{ij}(\tau), u^{ji}(\tau))$ is continuously differentiable in $x^{ij}(\tau)$, $u^{ij}(\tau)$ and $u^{ji}(\tau)$.

Define the payoff of each player i at each link or arc$i \Leftrightarrow j$ by

$$K_i^{ij}(x_0^{ij}, u^{ij}, u^{ji}, T - t_0) = \int_{t_0}^{T} h_i^j(x^{ij}(\tau), u^{ij}(\tau))d\tau.$$

Because player i plays multiple different differential games, the dynamic equation contains the player i's control and the control of his neighbor who plays the differential game with him. The payoff function of player i is not only dependent upon his control variable, which is from the control set $u^i(t) = (u^{ij}(t), j \in \widetilde{K}(i))$, and trajectories $x^i(t) = (x^{ij}(t), j \in \widetilde{K}(i))$ but also depend on the control variables of his neighbor, which is from the control set

$u^j(t) = (u^{ji}(t), i \in \widetilde{K}(j))$. Denote by $u(t) = (u^1(t), ..., u^i(t), ..., u^n(t))$, where $u^i(t) = (u^{ij}(t), j \in \widetilde{K}(i))$ is the control variable of player i in the network structure. We use $x_0 = (x_0^1, ..., x_0^i, ..., x_0^n)$ to denote the vector of initial conditions, here $x_0^i = (x^{ij}(t_0), j \in \widetilde{K}(i))$ is the set of initial conditions of player i.

The payoff function of player i is given by

$$H_i(x_0^i, u^i, u^j, T - t_0) = \sum_{j \in \widetilde{K}(i)} K_i^{ij}(x_0^{ij}, u^{ij}, u^{ji}, T - t_0) = \sum_{j \in \widetilde{K}(i)} \int\limits_{t_0}^{T} h_i^j(x^{ij}(\tau), u^{ij}(\tau)) d\tau. \quad (2)$$

Here, the term $h_i^j(x^{ij}(\tau), u^{ij}(\tau))$ is the instantaneous gain that player i can obtain through network links with player j. We also suppose that the term $h_i^j(x^{ij}(\tau), u^{ij}(\tau))$ is non-negative.

3. The characteristic function. The game $\Gamma(x_0, T - t_0)$ is defined on the network (N, L), the system dynamics (1) and players' payoffs are determined by (2). Player i ($i \in N$), choosing a control variable u^{ij} from his set of feasible controls, seeks to maximize his objective functional (2).

Suppose that players can cooperate to achieve the maximum total payoff:

$$\max_{u^1, ..., u^i, ..., u^n} \sum_{i \in N} \sum_{j \in \widetilde{K}(i)} \left(\int\limits_{t_0}^{T} h_i^j(x^{ij}(\tau), u^{ij}(\tau)) d\tau \right),$$

subject to dynamics (1) and the corresponding to the optimal cooperative strategies of players $\bar{u}(t) = (\bar{u}^1(t), ..., \bar{u}^i(t), ..., \bar{u}^l(t))$, where $\bar{u}^i(t) = (\bar{u}^{ij},\ j \in \widetilde{K}(i))$ exist. Denote the corresponding cooperative trajectory of player i by $\bar{x}^{ij}(t)$, $i \in N$, $j \in \widetilde{K}(i)$. The trajectory $\bar{x}(t) = (\bar{x}^1(t), ..., \bar{x}^i(t), ..., \bar{x}^n(t))$ is called optimal cooperative trajectory, where $\bar{x}^i(t) = (\bar{x}^{ij}(t),\ j \in \widetilde{K}(i))$.

Then the maximal joint payoff can be expressed as

$$\sum_{i \in N} \sum_{j \in \widetilde{K}(i)} \left(\int\limits_{t_0}^{T} h_i^j(\bar{x}^{ij}(\tau), \bar{u}^{ij}(\tau)) d\tau \right).$$

In [18] introduce a new characteristic function

$$V(S; x_0, T - t_0) = \sum_{i \in S} \sum_{j \in \widetilde{K}(i) \cap S} \left(\int\limits_{t_0}^{T} h_i^j(\bar{x}^{ij}(\tau), \bar{x}^{ji}(\tau)) d\tau \right) +$$

$$+ \alpha(S) \sum_{i \in S} \sum_{j \in \widetilde{K}(i) \cap N \setminus S} \left(\int\limits_{t_0}^{T} h_i^j(\bar{x}^{ij}(\tau), \bar{x}^{ji}(\tau)) d\tau \right)$$

for $S \subset N$. The values of characteristic function for each coalition are calculated as joint payoff of players from this coalition plus payoffs (multiplied on discount factor depending from S) of players which do not belong to the coalition S but have connections with players from S. We consider a special case of this characteristic function, when $\alpha(S) = \alpha$ and do not depend on coalition S, $\alpha \in [0, 1)$.

Defintion 1. The characteristic function $V(S;x_0,T-t_0)$ is defined as

$$V(S;x_0,T-t_0)=\sum_{i\in S}\sum_{j\in\widetilde{K}(i)\cap S}\left(\int\limits_{t_0}^{T}h_i^j(\bar{x}^{ij}(\tau),\bar{u}^{ij}(\tau))d\tau\right)+$$

$$+\,\alpha\sum_{i\in S}\sum_{j\in\widetilde{K}(i)\cap N\setminus S}\left(\int\limits_{t_0}^{T}h_i^j(\bar{x}^{ij}(\tau),\bar{u}^{ij}(\tau))d\tau\right) \tag{3}$$

for $S\subset N$. Here $\alpha\in[0,1)$, note that every player i from the coalition S plays the independent pairwise differential games with players j entering in $\widetilde{K}(i)\cap S$ and also with players outside coalition S (j belongs to the $\widetilde{K}(i)\cap(N\setminus S)$).

From (3), for coalitions $\{i\},\{\varnothing\},\{N\}$, we get equations

$$V(\{i\},x_0^i,T-t_0)=\alpha\sum_{j\in\widetilde{K}(i),j\neq i}\left(\int\limits_{t_0}^{T}h_i^j(\bar{x}^{ij}(\tau),\bar{u}^{ij}(\tau))d\tau\right), \tag{4}$$

$$V(\{\varnothing\},x_0,T-t_0)=0,$$

$$V(\{N\};x_0,T-t_0)=\max_{u^1,\dots,u^i,\dots,u^n}\sum_{i\in N}\sum_{j\in\widetilde{K}(i)}\left(\int\limits_{t_0}^{T}h_i^j(x^{ij}(\tau),u^{ij}(\tau))d\tau\right)=$$

$$=\sum_{i\in N}\sum_{j\in\widetilde{K}(i)}\left(\int\limits_{t_0}^{T}h_i^j(\bar{x}^{ij}(\tau),\bar{u}^{ij}(\tau))d\tau\right).$$

Defintion 2. The characteristic function $V(S;x_0,T-t_0)$ is called convex (or supermodular) if for any coalitions S_1, $S_2\subseteq N$ the following condition holds:

$$V(S_1\cup S_2;x_0,T-t_0)\geqslant V(S_1;x_0,T-t_0)+V(S_2;x_0,T-t_0)-V(S_1\cap S_2;x_0,T-t_0).$$

A game is called convex if its characteristic function is convex.

Proposition 1. The characteristic function $V(S;x_0,T-t_0)$ defined by formula (3), $S\subset N$ is convex.

P r o o f. See Appendix.

Proposition 2. The characteristic function $V(S;x_0,T-t_0)$ defined by formula (3), $S\subset N$ is time consistent.

From (4), we obtain

$$V(S;x_0,T-t_0)=\sum_{i\in S}\sum_{j\in\widetilde{K}(i)\cap S}\left(\int\limits_{t_0}^{t}h_i^j(\bar{x}^{ij}(\tau),\bar{u}^{ij}(\tau))d\tau\right)+$$

$$+\,\alpha\sum_{i\in S}\sum_{j\in\widetilde{K}(i)\cap N\setminus S}\left(\int\limits_{t_0}^{t}h_i^j(\bar{x}^{ij}(\tau),\bar{u}^{ij}(\tau))d\tau\right)+V(S;\bar{x}(t),T-t), \tag{5}$$

here $\alpha\in[0,1)$, and Equation on (5) show the time consistency property of the cooperative-trajectory characteristic function $V(S;x_0,T-t_0)$.

4. Cooperative game, the Shapley value. Next, we need to determine the rule for allocating the maximum total payoff between the players. This paper considers the core and the Shapley value as the optimality principles. We denote the set of all imputations as $\bar{L}(x_0, T - t_0)$

$$\bar{L}(x_0, T-t_0) = \{\xi(x_0, T-t_0) = (\xi_1(x_0, T-t_0), ..., \xi_n(x_0, T-t_0)) : V(N; x_0, T-t_0) =$$

$$= \sum_{i \in N} \xi_i(x_0, T-t_0), \xi_i(x_0, T-t_0) \geqslant V(\{i\}; x_0, T-t_0)\}$$

for $i \in N$.

4.1. The core.

Defintion 3. The core $\bar{C}(x_0, T-t_0)$ is the subset of imputations $\bar{L}(x_0, T-t_0)$, and is defined as

$$\bar{C}(x_0, T-t_0) = \{\xi(x_0, T-t_0) \in \bar{L}(x_0, T-t_0) : \sum_{i \in S} \xi_i(x_0, T-t_0) \geqslant V(S; x_0, T-t_0)\}$$

for $S \subset N$.

4.2. The Shapley value. Using the newly defined characteristic function, we introduce the Shapley value imputation in this subsection:

$$Sh_i(x_0, T-t_0) = \sum_{\substack{S \subset N \\ S \ni i}} \frac{(|S|-1)!(n-|S|)!}{n!} \times \tag{6}$$

$$\times [V(S; x_0, T-t_0) - V(S \setminus \{i\}; x_0, T-t_0)]$$

for $i \in N$. Form (6), we get

$$Sh_i(x_0, T-t_0) = \sum_{\substack{S \subset N \\ S \ni i}} \frac{(|S|-1)!(n-|S|)!}{n!} \times$$

$$\times \Bigg[\sum_{l \in S} \sum_{j \in \widetilde{K}(l) \cap S} \left(\int_{t_0}^{T} h_l^j(\bar{x}^{lj}(\tau), \bar{u}^{lj}(\tau)) d\tau \right) +$$

$$+ \alpha \sum_{l \in S} \sum_{j \in \widetilde{K}(l) \cap (N \setminus S)} \left(\int_{t_0}^{T} h_l^j(\bar{x}^{lj}(\tau), \bar{u}^{lj}(\tau)) d\tau \right) - \tag{7}$$

$$- \sum_{l \in S \setminus \{i\}} \sum_{j \in K(l) \cap S \setminus \{i\}} \left(\int_{t_0}^{T} h_l^j(\bar{x}^{lj}(\tau), \bar{u}^{lj}(\tau)) d\tau \right) -$$

$$- \alpha \sum_{l \in S \setminus \{i\}} \sum_{j \in \widetilde{K}(l) \cap N \setminus (S \setminus \{i\})} \left(\int_{t_0}^{T} h_l^j(\bar{x}^{lj}(\tau), \bar{u}^{lj}(\tau)) d\tau \right) \Bigg].$$

Theorem 1. *The Shapley value imputation in* (7) *satisfies the time consistency proporty.*

P r o o f. By direct computation we get formula

$$Sh_i(x_0, T-t_0) = \sum_{\substack{S\subset N\\ S\ni i}} \frac{(|S|-1)!(n-|S|)!}{n!} \times$$

$$\times \Bigg[\sum_{l\in S} \sum_{j\in \widetilde{K}(l)\cap S} \left(\int_{t_0}^{t} h_l^j(\bar{x}^{lj}(\tau), \bar{u}^{lj}(\tau))d\tau \right) +$$

$$+ \alpha \sum_{l\in S} \sum_{j\in \widetilde{K}(l)\cap N/S} \left(\int_{t_0}^{t} h_l^j(\bar{x}^{lj}(\tau), \bar{u}^{lj}(\tau))d\tau \right) -$$

$$- \sum_{l\in S\backslash\{i\}} \sum_{j\in \widetilde{K}(l)\cap S\backslash\{i\}} \left(\int_{t_0}^{t} h_l^j(\bar{x}^{lj}(\tau), \bar{u}^{lj}(\tau))d\tau \right) -$$

$$- \alpha \sum_{l\in S\backslash\{i\}} \sum_{j\in \widetilde{K}(i)\cap N\backslash(S\backslash\{i\})} \left(\int_{t_0}^{t} h_l^j(\bar{x}^{lj}(\tau), \bar{u}^{lj}(\tau))d\tau \right) \Bigg] + Sh_i(\bar{x}(t), T-t)$$

for $i \in N$, which exhibits the time consistency property of the Shapley value imputation, $Sh_i(\bar{x}(t), T-t)$ for $t \in [t_0, T]$.

In most cases, the Shapley value usually does not satisfy this condition [19–21].

5. Example. Consider following game-theoretic model of differential network games. The network structure is shown in Figure 1. There are three players to present the three national or regional factories that participate in the game with the network structure, $N = \{1, 2, 3\}$.

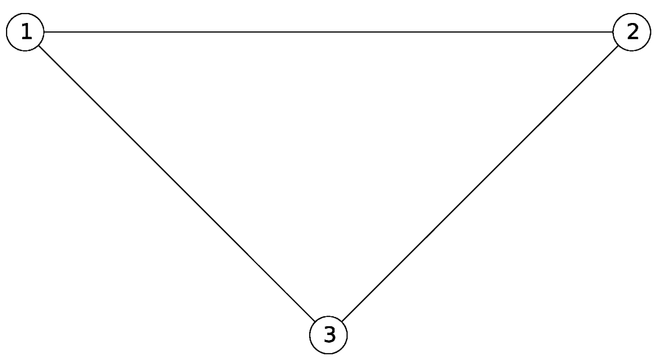

Figure 1. Network structure

As for arc$(1,2)$ (similar game is considered in [22]). Regions 1 and 2 play the pollution game. Each region has an industrial production site. The production is assumed to be proportional to the pollution u^{12} and u^{21}. Thus the strategy of each player is to choose the amount of pollutants emitted to the atmosphere, $u^{12} \in [0, b_{12}]$, $b_{12} > 0$, $u^{21} \in [0, b_{21}]$, $b_{21} > 0$, A_{12} is the amount that the government subsidizes to the factory 1 at each moment, $d_{12}x^{12}(t)$ is the environment department that penalizes factory 1 at each moment. The dynamics of each players 1 and 2 on arc$(1,2)$ is described by

$$\dot{x}^{12}(t) = u^{12}(t) + u^{21}(t), \quad x^{12}(t_0) = x_0^{12}, \quad t \in [t_0, T], \tag{8}$$

$$\dot{x}^{21}(t) = u^{21}(t) + u^{12}(t), \quad x^{21}(t_0) = x_0^{21}, \quad t \in [t_0, T]. \tag{9}$$

The payoff of each player in the pairwise interactions game on the arc$(1, 2)$ is defined as

$$K_1^{12}(x_0^{12}, u^{12}(t), u^{21}(t), T - t_0) = \int_{t_0}^{T} \left[\left(b_{12} - \frac{1}{2} u^{12}(t) \right) u^{12}(t) - d_{12} x^{12}(t) + A_{12} \right] dt,$$

$$K_2^{21}(x_0^{21}, u^{12}(t), u^{21}(t), T - t_0) = \int_{t_0}^{T} \left[\left(b_{21} - \frac{1}{2} u^{21}(t) \right) u^{21}(t) - d_{21} x^{21}(t) + A_{21} \right] dt.$$

As for pair arc$(1, 3)$ (similar game is considered in [23]), we examiner another pollution game. The release pollution of each player 1 or 3 are denoted by u^{13} and u^{31}, where $u^{13} \in [0, b_{13}]$, $b_{13} > 0$, $u^{31} \in [0, b_{31}]$, $b_{31} > 0$. Let $x^{13}(t)$ and $x^{31}(t)$ denote the stock of accumulated pollution by time t. The dynamics of each player 1 and 3 at pair arc$(1, 3)$ is described by

$$\dot{x}^{13}(t) = u^{13}(t) + u^{31}(t) - \delta x^{13}, \quad x^{13}(t_0) = x_0^{13}, \quad t \in [t_0, T]. \tag{10}$$

$$\dot{x}^{31}(t) = u^{13}(t) + u^{31}(t) - \delta x^{31}, \quad x^{31}(t_0) = x_0^{31}, \quad t \in [t_0, T]. \tag{11}$$

Where δ is the absorption coefficient corresponding to the natural purification of the atmosphere, we assume that $\delta > 0$. Here we don't consider the additional cost. The payoff of each player in the pairwise interactions game on arc$(1, 3)$ is defined as

$$K_1^{13}(x_0^{13}, u^{13}(t), u^{31}(t), T - t_0) = \int_{t_0}^{T} \left(\left(b_{13} - \frac{1}{2} u^{13}(t) \right) u^{13}(t) - d_{13} x^{13}(t) + A_{13} \right) dt,$$

$$K_3^{31}(x_0^{31}, u^{13}(t), u^{31}(t), T - t_0) = \int_{t_0}^{T} \left(\left(b_{31} - \frac{1}{2} u^{31}(t) \right) u^{31}(t) - d_{31} x^{31}(t) + A_{31} \right) dt.$$

As for arc$(2, 3)$ (similar game is considered in [24]), we examiner another pollution game. The dynamics of the stock of pollution for each player at arc$(2, 3)$ is described by

$$\dot{x}^{23}(t) = \mu(u^{23}(t) + u^{32}(t)) - \epsilon x^{23}(t), \quad x^{23}(t_0) = x_0^{23}, \quad t \in [t_0, T]. \tag{12}$$

Here $\mu > 0$ is the marginal influence on pollution accumulation x^{23} issued by the players' emissions, and $\epsilon > 0$, $\epsilon \neq \delta$ is the rate of natural absorption:

$$\dot{x}^{32}(t) = \mu(u^{32}(t) + u^{23}(t)) - \epsilon x^{32}(t), \quad x^{32}(t_0) = x_0^{32}, \quad t \in [t_0, T]. \tag{13}$$

The payoff of each player at arc$(2, 3)$ is defined as

$$K_2^{23}(x_0^{23}, u^{23}(t), u^{32}(t), T - t_0) = \int_{t_0}^{T} \left[\left(b_{23} - \frac{1}{2} u^{23}(t) \right) u^{23}(t) - d_{23} x^{23}(t) + A_{23} \right] dt,$$

$$K_3^{32}(x_0^{32}, u^{23}(t), u^{32}(t), T-t_0) = \int_{t_0}^{T} \left[\left(b_{32} - \frac{1}{2} u^{32}(t) \right) u^{32}(t) - d_{32} x^{32}(t) + A_{32} \right] dt.$$

In the network game, as for multiple links, the payoff of each player is defined as

$$H_1(x_0^1, u^{12}(t), u^{21}(t), u^{13}(t), u^{31}(t), T-t_0) = \int_{t_0}^{T} \left[\left(b_{12} - \frac{1}{2} u^{12}(t) \right) u^{12}(t) - \right.$$

$$\left. - d_{12} x^{12}(t) + A_{12} \right] dt + \int_{t_0}^{T} \left[\left(b_{13} - \frac{1}{2} u^{13}(t) \right) u^{13}(t) - d_{13} x^{13}(t) + A_{13} \right] dt,$$

$$H_2(x_0^2, u^{12}(t), u^{21}(t), u^{23}(t), u^{32}(t), T-t_0) = \int_{t_0}^{T} \left[\left(b_{21} - \frac{1}{2} u^{21}(t) \right) u^{21}(t) - \right.$$

$$\left. - d_{21} x^{21}(t) + A_{21} \right] dt + \int_{t_0}^{T} \left[\left(b_{23} - \frac{1}{2} u^{23}(t) \right) u^{23}(t) - d_{23} x^{23}(t) + A_{23} \right] dt,$$

$$H_3(x_0^3, u^{13}(t), u^{31}(t), u^{23}(t), u^{32}(t), T-t_0) = \int_{t_0}^{T} \left[\left(b_{31} - \frac{1}{2} u^{31}(t) \right) u^{31}(t) - \right.$$

$$\left. - d_{31} x^{31}(t) + A_{31} \right] dt + \int_{t_0}^{T} \left[\left(b_{32} - \frac{1}{2} u^{32}(t) \right) u^{32}(t) - d_{32} x^{32}(t) + A_{32} \right] dt.$$

Subject to dynamics (8)–(13).

Under the cooperation, players maximize the total payoff

$$V(\{N\}; x_0, T-t_0) =$$

$$= \max_{u^{12}, u^{21}, u^{13}, u^{31}, u^{32}, u^{23}} \sum_{i \in N} \sum_{j \in \widetilde{K}(i)} \int_{t_0}^{T} \left[\left(b_{ij} - \frac{1}{2} u^{ij} \right) u^{ij} - d_{ij} x^{ij} + A_{ij} \right] dt.$$

Using Pontryagin Maximum Principle (PMP) to solve the optimization problem, firstly, write down the Hamiltonian function:

$$H(x_0, T-t_0, u(t), \varphi) = \sum_{i \in 1,2,3} \sum_{j \in \widetilde{K}(i)} [(b_{ij} - \frac{1}{2} u^{ij}) u^{ij} - d_{ij} x^{ij} + A_{ij}] + \varphi_{12}(u^{12} + u^{21}) +$$

$$+ \varphi_{21}(u^{21} + u^{12}) + \varphi_{13}(u^{13} + u^{31} - \delta x^{13}) + \varphi_{31}(u^{31} + u^{13} - \delta x^{31}) +$$
$$+ \varphi_{23}(\mu(u^{23} + u^{32}) - \epsilon x^{23}) + \varphi_{32}(\mu(u^{32} + u^{23}) - \epsilon x^{32}).$$

Here we have the following boundary conditions on adjoint variable $\varphi_{ij}(t)$:

$$\varphi_{ij}(T) = 0.$$

Taking the first derivative with respect to u^{12}, we get the expressions for the optimal controls:

$$\bar{u}^{12}(t) = b_{12} + (\varphi_{12} + \varphi_{21}).$$

The canonical system is written as

$$\dot{x}^{12} = u^{12} + u^{21} = b + 2(\varphi_{12} + \varphi_{21}), \tag{14}$$

$$\dot{\varphi}_{12} = d_{12}, \quad \dot{\varphi}_{21} = d_{21},$$

where $b = b_{12} + b_{21}$.

Recall that the initial condition is $x^{12}(t_0) = x_0^{12}$, also using another boundary condition, which is obtained from (8)–(13), then we get the

$$\varphi_{12}(t) = -d_{12}(T - t),$$

$$\varphi_{21}(t) = -d_{21}(T - t).$$

Substitute this solution to the differential equation (14) to obtain the expression for $\bar{x}^{12}(t)$:

$$\bar{x}^{12}(t) = dt^2 - dt_0^2 + (b - 2dT)t + (-b + 2Td)t_0 + x_0^{12},$$

here $d = d_{12} + d_{21}$. The optimal control is

$$\bar{u}^{12}(t) = b_{12} - d(T - t).$$

Similarly, we get the optimal trajectories:

$$\bar{x}^{21}(t) = dt^2 - dt_0^2 + (b - 2dT)t + (-b + 2Td)t_0 + x_0^{21},$$

$$\bar{x}^{13}(t) = C_{13}e^{-\delta t} + \frac{\bar{b}}{\delta} - \frac{e^{-\delta(T-t)}\bar{d}}{\delta^2},$$

here $C_{13} = e^{\delta t_0}(x_0^{13} - \frac{\bar{b}}{\delta} + \frac{e^{-\delta(T-t_0)}\bar{d}}{\delta})$, $\bar{b} = b_{13} + b_{31}$, $\bar{d} = d_{13} + d_{31}$;

$$\bar{x}^{31}(t) = C_{31}e^{-\delta t} + \frac{\bar{b}}{\delta} - \frac{e^{-\delta(T-t)}\bar{d}}{\delta^2},$$

here $C_{31} = e^{\delta t_0}(x_0^{31} - \frac{\bar{b}}{\delta} + \frac{e^{-\delta(T-t_0)}\bar{d}}{\delta})$;

$$\bar{x}^{23}(t) = C_{23}e^{-\epsilon t} + \frac{\mu\hat{b}}{\epsilon} + \frac{e^{-\epsilon(T-t)}\hat{d}\mu^2}{\epsilon^2} - \frac{2\hat{d}\mu^2}{\epsilon^2},$$

here $C_{23} = e^{\epsilon t_0}(x_0^{23} - \frac{\mu\hat{b}}{\epsilon} + \frac{e^{-\epsilon(T-t_0)}\hat{d}\mu^2}{\epsilon^2} + \frac{2\hat{d}\mu^2}{\epsilon^2})$, $\hat{d} = d_{23} + d_{32}$, $\hat{b} = b_{23} + b_{32}$;

$$\bar{x}^{32}(t) = C_{32}e^{-\epsilon t} + \frac{\mu\hat{b}}{\epsilon} + \frac{e^{-\epsilon(T-t)}\hat{d}\mu^2}{\epsilon^2} - \frac{2\hat{d}\mu^2}{\epsilon^2},$$

here $C_{32} = e^{\epsilon t_0}(x_0^{32} - \frac{\mu\hat{b}}{\epsilon} + \frac{e^{-\epsilon(T-t_0)}\hat{d}\mu^2}{\epsilon^2} + \frac{2\hat{d}\mu^2}{\epsilon^2})$, $\hat{d} = d_{23} + d_{32}$, $\hat{b} = b_{23} + b_{32}$. The corresponding optimal controls are

$$\bar{u}^{21}(t) = b_{21} - d(T - t),$$

$$\bar{u}^{13}(t) = b_{13} - \frac{e^{-\delta(T-t)}\bar{d}}{\delta},$$

$$\bar{u}^{31}(t) = b_{31} - \frac{e^{-\delta(T-t)}\bar{d}}{\delta},$$

$$\bar{u}^{23}(t) = b_{23} - \frac{\mu \cdot \hat{d}e^{-\epsilon(T-t)}}{\epsilon},$$

$$\bar{u}^{32}(t) = b_{32} - \frac{\mu\hat{d}e^{-\epsilon(T-t)}}{\epsilon},$$

$$V(\{1\}, x_0, T-t_0) = \alpha \left(\int\limits_{t_0}^{T} \left[\left(b_{12} - \frac{1}{2}\bar{u}^{12}(t) \right) \bar{u}^{12}(t) - d_{12}\bar{x}^{12}(t) \; + \right.\right.$$

$$\left.\left. + \; A_{12} \right] dt + \int\limits_{t_0}^{T} \left[\left(b_{13} - \frac{1}{2}\bar{u}^{13}(t) \right) \bar{u}^{13}(t) - d_{13}\bar{x}^{13}(t) + A_{13} \right] dt \right),$$

$$V(\{2\}, x_0, T-t_0) = \alpha \left(\int\limits_{t_0}^{T} \left[\left(b_{21} - \frac{1}{2}\bar{u}^{21}(t) \right) \bar{u}^{21}(t) - d_{21}\bar{x}^{21}(t) \; + \right.\right.$$

$$\left.\left. + \; A_{21} \right] dt + \int\limits_{t_0}^{T} \left[\left(b_{23} - \frac{1}{2}\bar{u}^{23}(t) \right) \bar{u}^{23}(t) - d_{23}\bar{x}^{23}(T) + A_{23} \right] dt \right),$$

$$V(\{3\}, x_0, T-t_0) = \alpha \left(\int\limits_{t_0}^{T} \left[\left(b_{31} - \frac{1}{2}\bar{u}^{31}(t) \right) \bar{u}^{31}(t) - d_{31}\bar{x}^{31}(t) \; + \right.\right.$$

$$\left.\left. + \; A_{31} \right] dt + \int\limits_{t_0}^{T} \left[\left(b_{32} - \bar{u}^{32}(t) \right) \bar{u}^{32}(t) - d_{32}\bar{x}^{32}(t) + A_{32} \right] dt \right),$$

$$V(\{1,2\}, x_0, T-t_0) = \int\limits_{t_0}^{T} \left[\left(b_{12} - \frac{1}{2}\bar{u}^{12}(t) \right) \bar{u}^{12}(t) - d_{12}\bar{x}^{12} + A_{12} \right] dt +$$

$$+ \int\limits_{t_0}^{T} \left[\left(b_{21} - \frac{1}{2}\bar{u}^{21}(t) \right) \bar{u}^{21}(t) - d_{21}\bar{x}^{21}(t) + A_{21} \right] dt + \alpha \left(\int\limits_{t_0}^{T} \left[\left(b_{13} - \frac{1}{2}\bar{u}^{13}(t) \right) \bar{u}^{13}(t) - \right.\right.$$

$$\left.\left. - \; d_{13}\bar{x}^{13}(t) + A_{13} \right] dt \right) + \alpha \left(\int\limits_{t_0}^{T} \left[(b_{23} - \frac{1}{2}\bar{u}^{23})\bar{u}^{23}(t) - d_{23}x^{23}(t) + A_{23} \right] dt \right),$$

$$V(\{1,3\}, x_0, T-t_0) = \int\limits_{t_0}^{T} \left[\left(b_{13} - \frac{1}{2}\bar{u}^{13}(t)\right)\bar{u}^{13}(t) - d_{13}\bar{x}^{13}(t) + A_{13}\right] dt +$$

$$+ \int\limits_{t_0}^{T} \left[\left(b_{31} - \frac{1}{2}\bar{u}^{31}(t)\right)\bar{u}^{31}(t) - d_{31}\bar{x}^{31}(t) + A_{31}\right] dt + \alpha\left(\int\limits_{t_0}^{T} \left[\left(b_{12} - \frac{1}{2}\bar{u}^{12}(t)\right)\bar{u}^{12}(t) -\right.\right.$$

$$\left.\left. - d_{12}\bar{x}^{12}(t) + A_{12}\right] dt\right) + \alpha\left(\int\limits_{t_0}^{T} \left[\left(b_{32} - \frac{1}{2}\bar{u}^{32}\right)\bar{u}^{32} - d_{32}\bar{x}^{32}(t) + A_{32}\right] dt\right),$$

$$V(\{2,3\}, x_0, T-t_0) = \int\limits_{t_0}^{T} [(b_{23} - \bar{u}^{23}(t))\bar{u}^{23}(t) - d_{23}\bar{x}^{23}(t) + A_{23}]dt +$$

$$+ \int\limits_{t_0}^{T} \left[\left(b_{32} - \bar{u}^{32}(t)\right)\bar{u}^{32}(t) - d_{32}\bar{x}^{32}(t) + A_{32}\right] dt + \alpha\left(\int\limits_{t_0}^{T} \left[\left(b_{21} - \frac{1}{2}\bar{u}^{21}(t)\right)\bar{u}^{21}(t) -\right.\right.$$

$$\left.\left. - d_{21}\bar{x}^{21}(t) + A_{21}\right] dt\right) + \alpha\left(\int\limits_{t_0}^{T} \left[\left(b_{31} - \frac{1}{2}\bar{u}^{31}(t)\right)\bar{u}^{31}(t) - d_{31}\bar{x}^{31}(t) + A_{31}\right] dt\right).$$

Remark. The instantaneous payoff in the game is $(b_{ij} - \frac{1}{2}u^{ij}(t))u^{ij}(t) - d_{ij}x^{ij}(t) + A_{ij}$, since $(b_{ij} - \frac{1}{2}u^{ij}(t))u^{ij}(t) \geqslant 0$, $u^{ij} \in [0, b_{ij}]$, if $A_{ij} \geqslant \max_{x^{ij}(t)}(d_{ij}x^{ij}(t))$, $t \in [t_0, T]$, then all instantaneous payoff for each player at any time t are non-negative. Because of $u^{ij}(t) \in [0, b_{ij}]$, $u^{ji} \in [0, b_{ji}]$, if we treat it as a constant and integrate the differential equations (8)–(13), we obtain:

$$\begin{cases} x^{12}(t) = (u^{12} + u^{21})(t - t_0) + x_0^{12}, \\ x^{21}(t) = (u^{12} + u^{21})(t - t_0) + x_0^{21}, \end{cases}$$

$$\begin{cases} x^{13}(t) = \frac{(u^{13}+u^{31})}{\delta}(1 - e^{-\delta(t-t_0)}) + e^{-\delta(t-t_0)}x_0^{13}, \\ x^{31}(t) = \frac{(u^{13}+u^{31})}{\delta}(1 - e^{-\delta(t-t_0)}) + e^{-\delta(t-t_0)}x_0^{31}, \end{cases}$$

$$\begin{cases} x^{23}(t) = e^{-\epsilon(t-t_0)}x_0^{23} + \frac{\mu(u^{23}+u^{32})}{\epsilon}(1 - e^{-\epsilon(t-t_0)}), \\ x^{32}(t) = e^{-\epsilon(t-t_0)}x_0^{32} + \frac{(u^{23}+u^{32})\mu}{\epsilon}(1 - e^{-\epsilon(t-t_0)}). \end{cases}$$

Additional conditions

$$A_{12} \geqslant \max_{x^{12}}(d_{12}x^{12}(t)) = d_{12}[b(T - t_0) + x_0^{12}],$$

$$A_{21} \geqslant \max_{x^{21}}(d_{21}x^{21}(t)) = d_{12}[b(T - t_0) + x_0^{21}],$$

$$A_{13} \geqslant \max_{x^{13}}(d_{13}x^{13}(t)) = d_{13}[x_0^{13}e^{-\delta(T-t_0)} + \frac{\bar{b}}{\delta}(1 - e^{-\delta(T-t_0)})],$$

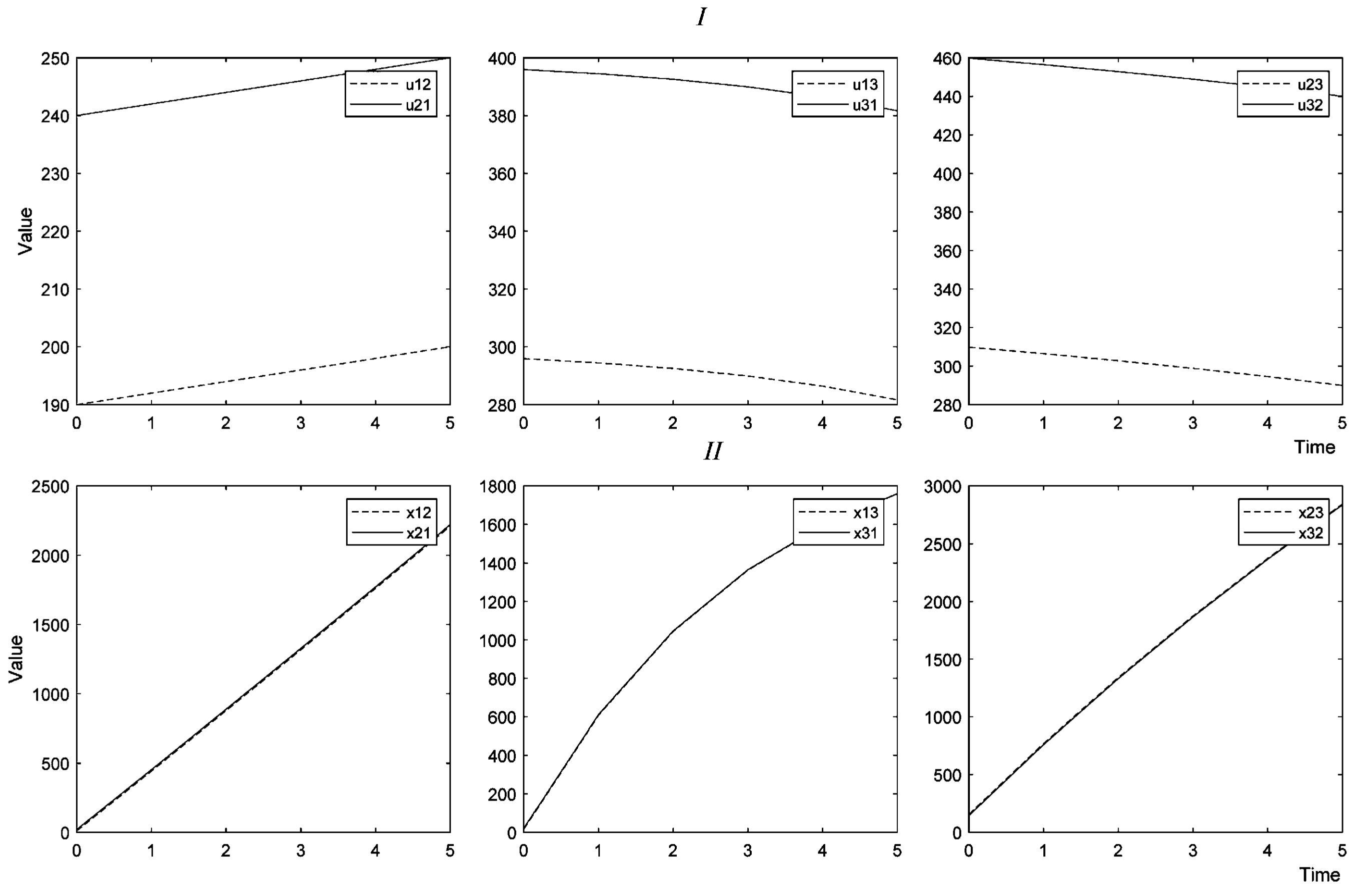

Figure 2. Optimal control variables (I) and optimal trajectories (II) in the pairwise interaction games for each player

$$A_{31} \geqslant \max_{x^{31}}(d_{31}x^{31}(t)) = d_{31}[x_0^{31}e^{-\delta(T-t_0)} + \frac{\bar{b}}{\delta}(1 - e^{-\delta(T-t_0)})],$$

$$A_{23} \geqslant \max_{x^{23}}(d_{23}x^{23}(t)) = d_{23}[x_0^{23}e^{-\epsilon(T-t_0)} + \frac{\mu\hat{b}}{\epsilon}(1 - e^{-\epsilon(T-t_0)})],$$

$$A_{32} \geqslant \max_{x^{32}}(d_{32}x^{32}(t)) = d_{32}[x_0^{32}e^{-\epsilon(T-t_0)} + \frac{\mu\hat{b}}{\epsilon}(1 - e^{-\epsilon(T-t_0)})].$$

Try to compute the core and the Shapley value. Assume the following values of the parameters: $b_{12} = 200, b_{21} = 250, b_{13} = 300, b_{31} = 400, b_{23} = 350, b_{32} = 500, d_{12} = 1, d_{21} = 1.5, d_{13} = 2.5, d_{31} = 3, d_{23} = 2.5, d_{32} = 3.5, \delta = 0.3, \mu = 0.8, \epsilon = 0.08, \alpha = 0.1, t_0 = 0, t = 2.5, T = 5, x_0^{12} = 10, x_0^{21} = 20, x_0^{13} = 25, x_0^{31} = 30, x_0^{23} = 50, x_0^{32} = 40$, $A_{12} = 2260, A_{21} = 3405, A_{13} = 4545.69, A_{31} = 5458.17, A_{23} = 7089.5, A_{32} = 9901.535$, $V(N, x_0, T - t_0) = 1.975334 \cdot 10^6$. Then we calculate

$$V(\{1\}, x_0, T - t_0) = 3.383075 \cdot 10^4,$$

$$V(\{2\}, x_0, T - t_0) = 4.807020 \cdot 10^4,$$

$$V(\{3\}, x_0, T - t_0) = 1.156325 \cdot 10^5,$$

$$V(\{1, 2\}, x_0, T - t_0) = 3.254166 \cdot 10^5,$$

$$V(\{1, 3\}, x_0, T - t_0) = 8.226729 \cdot 10^5,$$

$$V(\{2, 3\}, x_0, T - t_0) = 1.024778 \cdot 10^6,$$

$$Sh(x_0; T - t_0) = (4.921934 \cdot 10^5, 6.003659 \cdot 10^5, 8.827752 \cdot 10^5).$$

The numerical results are displayed in Figures 2–5. In Figure 2, we plotted the optimal policy $\bar{u}^{ij}$, and optimal trajectories $\bar{x}^{ij}$, $i \in 1, 2, 3$. Given three graphic interpretations of the obtained results. Figures 3–5 show the domains corresponding to the feasible imputation set $\bar{L}(x_0, T - t_0)$, and the core $\bar{C}(x_0, T - t_0)$ constructed using $V(S, x_0, T - t_0)$. Color with shadows represents the core, and the blank star represents the Shapley value imputation. The Figure 3 represents the game on the time interval form t_0 to T. On Figure 4 is the subgame that happened on the time interval $[t_0, t]$, and on Figure 5 is the subgame that happened on the time interval $[t, T]$. In our case, we use t_0, t, T as above value. The resulting Shapley value imputation belongs to the core of the initial game.

Table. **Values of characteristic function**

S	$V(S, x_0, t - t_0)$	$V(S, \bar{x}(t), T - t)$	$V(S, x_0, T - t_0)$
N	$1.003462 \cdot 10^6$	$9.941128 \cdot 10^5$	$1.975334 \cdot 10^6$
$\{1\}$	$1.733632 \cdot 10^4$	$1.787801 \cdot 10^4$	$1.649443 \cdot 10^4$
$\{2\}$	$2.448433 \cdot 10^4$	$2.358587 \cdot 10^4$	$4.807020 \cdot 10^4$
$\{3\}$	$5.852549 \cdot 10^4$	$5.710699 \cdot 10^4$	$1.156325 \cdot 10^5$
$\{1, 2\}$	$1.666546 \cdot 10^5$	$1.587619 \cdot 10^5$	$3.254166 \cdot 10^5$
$\{1, 3\}$	$4.180839 \cdot 10^5$	$4.045889 \cdot 10^5$	$8.226729 \cdot 10^5$
$\{2, 3\}$	$5.190691 \cdot 10^5$	$5.057093 \cdot 10^5$	$1.024778 \cdot 10^6$

To illustrate the time consistency, we choose the Shapley value as the cooperative solution. Then the payoffs of players at time period $[0, t]$ are $(2.508643 \cdot 10^5, 3.049309 \times 10^5, 4.476661 \cdot 10^5)$, and at the time period $[t, T]$ are $(2.584275 \cdot 10^5, 2.954349 \cdot 10^5, 4.351089 \times$

10^5). Furthermore, from the Table, the values of the characteristic function is also time consistent.

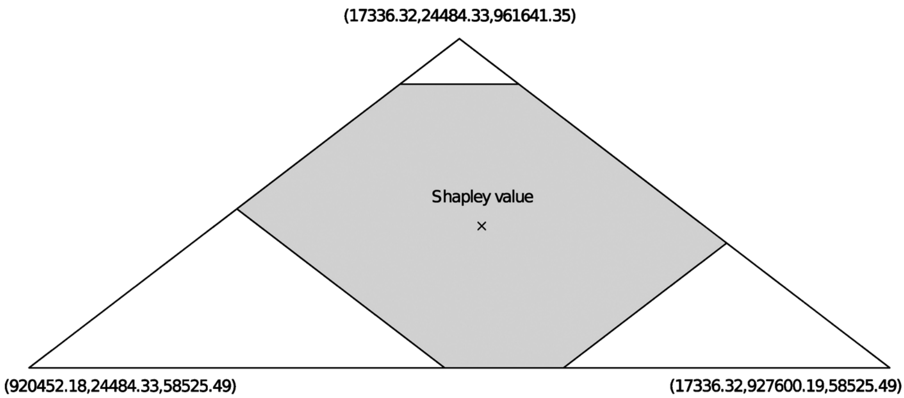

Figure 3. Cooperative game at $[t_0, T]$ (× is Shapley value also for Figures 4, 5)

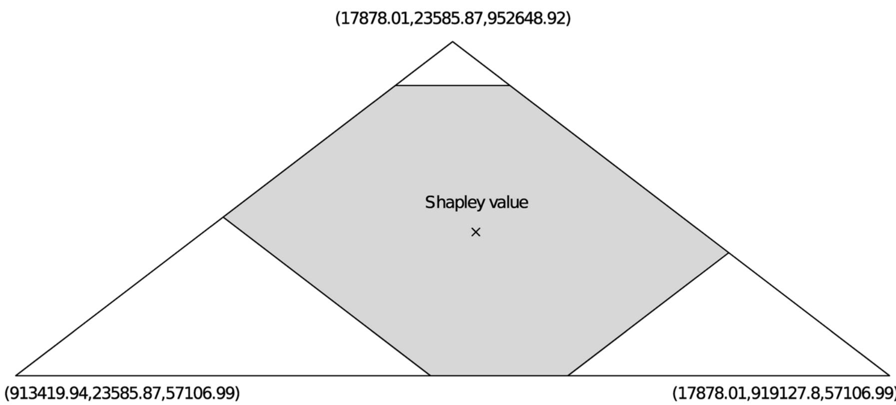

Figure 4. Cooperative game at $[t_0, t]$

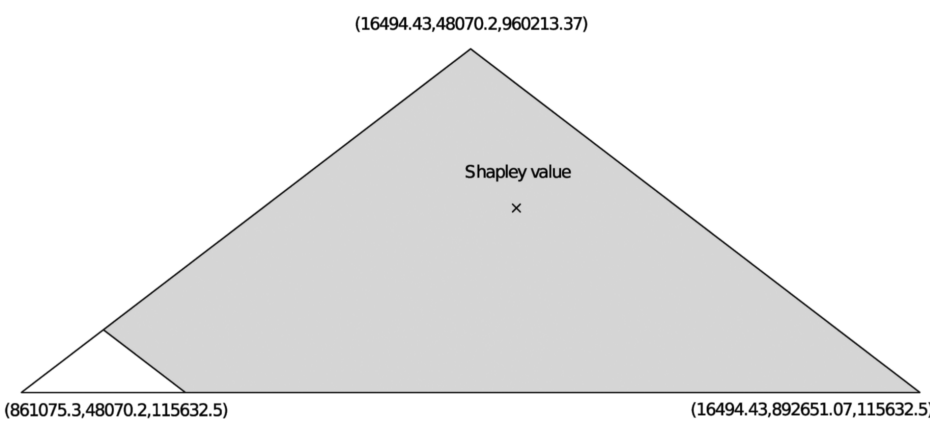

Figure 5. Cooperative game at $[t, T]$

6. Conclusion. In this paper, we studied the differential game with pairwise interaction, a new type of game in game theory. This give us the possibility to getting the new characteristic function in the game. The convexity of characteristic function is proved. By cooperation, we considered the Shapley value and the core as solutions. The key of this research is the differential game with pairwise interactions, where each player can play multiple different differential games. Finally, the results are illustrated by an example.

Appendix. P r o o f of proposition.

We also introduce the additional notation

$$w_{ij} = \int\limits_{t_0}^{T} h_i^j(\bar{x}^{ij}(\tau), \bar{u}^{ij}(\tau))d\tau.$$

Using (4), (5), we can rewrite:

$$v(S_1 \cup S_2; x_0, T - t_0) = \sum_{\substack{i \in S_1 \cup S_2 \\ j \in \widetilde{K}(i) \cap (S_1 \cup S_2)}} w_{ij} + \alpha \sum_{\substack{i \in S_1 \cup S_2 \\ j \in \widetilde{K}(i) \cap (N \setminus (S_1 \cup S_2))}} w_{ij} =$$

$$= \sum_{\substack{i \in S_1 \setminus S_2 \\ j \in \widetilde{K}(i) \cap (S_1 \setminus S_2)}} w_{ij} + \sum_{\substack{i \in S_1 \setminus S_2 \\ j \in \widetilde{K}(i) \cap (S_2 \setminus S_1)}} w_{ij} + \sum_{\substack{i \in S_1 \setminus S_2 \\ j \in \widetilde{K}(i) \cap (S_1 \cap S_2)}} w_{ij} + \sum_{\substack{i \in S_2 \setminus S_1 \\ j \in \widetilde{K}(i) \cap (S_2 \cap S_1)}} w_{ij} +$$

$$+ \sum_{\substack{i \in S_2 \setminus S_1 \\ j \in \widetilde{K}(i) \cap (S_2 \setminus S_1)}} w_{ij} + \sum_{\substack{i \in S_2 \setminus S_1 \\ j \in \widetilde{K}(i) \cap (S_1 \setminus S_2)}} w_{ij} + \sum_{\substack{i \in S_1 \cap S_2 \\ j \in \widetilde{K}(i) \cap (S_1 \setminus S_2)}} w_{ij} + \sum_{\substack{i \in S_1 \cap S_2 \\ j \in \widetilde{K}(i) \cap (S_1 \cap S_2)}} w_{ij} +$$

$$+ \sum_{\substack{i \in S_1 \cap S_2 \\ j \in \widetilde{K}(i) \cap (S_2 \setminus S_1)}} w_{ij} + \alpha \sum_{\substack{i \in S_1 \cap S_2 \\ j \in \widetilde{K}(i) \cap (N \setminus (S_1 \cup S_2))}} w_{ij} + \alpha \sum_{\substack{i \in S_1 \setminus S_2 \\ j \in \widetilde{K}(i) \cap (N \setminus (S_1 \cup S_2))}} w_{ij} +$$

$$+ \alpha \sum_{\substack{i \in S_2 \setminus S_1 \\ j \in \widetilde{K}(i) \cap (N \setminus (S_1 \cup S_2))}} w_{ij}, \tag{15}$$

$$v(S_1; x_0, T - t_0) = \sum_{\substack{i \in S_1 \\ j \in \widetilde{K}(i) \cap S_1}} w_{ij} + \alpha \sum_{\substack{i \in S_1 \\ j \in \widetilde{K}(i) \cap (N \setminus S_1)}} w_{ij} = \sum_{\substack{i \in S_1 \setminus S_2 \\ j \in \widetilde{K}(i) \cap (S_1 \cap S_2)}} w_{ij} +$$

$$+ \sum_{\substack{i \in S_1 \setminus S_2 \\ j \in \widetilde{K}(i) \cap (S_1 \setminus S_2)}} w_{ij} + \sum_{\substack{i \in S_1 \cap S_2 \\ j \in \widetilde{K}(i) \cap (S_1 \cap S_2)}} w_{ij} + \sum_{\substack{i \in S_1 \cap S_2 \\ j \in \widetilde{K}(i) \cap (S_1 \setminus S_2)}} w_{ij} + \alpha \sum_{\substack{i \in S_1 \setminus S_2 \\ j \in \widetilde{K}(i) \cap (N \setminus (S_1 \cup S_2))}} w_{ij} +$$

$$+ \alpha \sum_{\substack{i \in S_1 \setminus S_2 \\ j \in \widetilde{K}(i) \cap (S_2 \setminus S_1)}} w_{ij} + \alpha \sum_{\substack{i \in S_1 \cap S_2 \\ j \in \widetilde{K}(i) \cap (N \setminus (S_1 \cup S_2))}} w_{ij} + \alpha \sum_{\substack{i \in S_1 \cap S_2 \\ j \in \widetilde{K}(i) \cap (S_2 \setminus S_1)}} w_{ij}, \tag{16}$$

$$v(S_2; x_0, T - t_0) = \sum_{\substack{i \in S_2 \\ j \in \widetilde{K}(i) \cap S_2}} w_{ij} + \alpha \sum_{\substack{i \in S_2 \\ j \in \widetilde{K}(i) \cap (N \setminus S_2)}} w_{ij} = \sum_{\substack{i \in S_2 \setminus S_1 \\ j \in \widetilde{K}(i) \cap (S_2 \setminus S_1)}} w_{ij} +$$

$$+ \sum_{\substack{i \in S_2 \setminus S_1 \\ j \in \widetilde{K}(i) \cap (S_2 \cap S_1)}} w_{ij} + \sum_{\substack{i \in S_2 \cap S_1 \\ j \in \widetilde{K}(i) \cap (S_2 \cap S_1)}} w_{ij} + \sum_{\substack{i \in S_2 \cap S_1 \\ j \in \widetilde{K}(i) \cap (S_2 \setminus S_1)}} w_{ij} + \alpha \sum_{\substack{i \in S_2 \setminus S_1 \\ j \in \widetilde{K}(i) \cap (N \setminus (S_2 \cup S_1))}} w_{ij} +$$

$$+ \alpha \sum_{\substack{i\in S_2\setminus S_1\\ j\in \widetilde{K}(i)\cap(S_1\setminus S_2)}} w_{ij} + \alpha \sum_{\substack{i\in S_2\cap S_1\\ j\in \widetilde{K}(i)\cap(N\setminus(S_2\cup S_1))}} w_{ij} + \alpha \sum_{\substack{i\in S_2\cap S_1\\ j\in \widetilde{K}(i)\cap(S_1\setminus S_2)}} w_{ij}, \qquad (17)$$

$$v(S_1\cap S_2; x_0, T-t_0) = \sum_{\substack{i\in S_1\cap S_2\\ j\in \widetilde{K}(i)\cap(S_1\cap S_2)}} w_{ij} + \alpha \sum_{\substack{i\in S_1\cap S_2\\ j\in \widetilde{K}(i)\cap(N\setminus(S_1\cap S_2))}} w_{ij} =$$

$$= \sum_{\substack{i\in S_1\cap S_2\\ j\in \widetilde{K}(i)\cap(S_1\cap S_2)}} w_{ij} + \alpha \sum_{\substack{i\in S_1\cap S_2\\ j\in \widetilde{K}(i)\cap(N\setminus(S_1\cup S_2))}} w_{ij} + \alpha \sum_{\substack{i\in S_1\cap S_2\\ j\in \widetilde{K}(i)\cap(S_1\setminus S_2)}} w_{ij} +$$

$$+ \alpha \sum_{\substack{i\in S_1\cap S_2\\ j\in \widetilde{K}(i)\cap(S_2\setminus S_1)}} w_{ij}. \qquad (18)$$

Subtracting the expressions (16), (17) from (15) and adding (18), we obtain formula

$$(1-\alpha)\left(\sum_{\substack{i\in S_1\setminus S_2\\ j\in \widetilde{K}(i)\cap(S_2\setminus S_1)}} w_{ij} + \sum_{\substack{i\in S_2\setminus S_1\\ j\in \widetilde{K}(i)\cap(S_1\setminus S_2)}} w_{ij} \right) \geqslant 0.$$

The inequality follows from the non-negativity of payoffs. The statement of the proposition is proved.

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Received: November 20, 2023.
Accepted: December 26, 2023.

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Заметка о кооперативных дифференциальных играх с парными взаимодействиями*

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* Исследование выполнено за счет гранта Российского научного фонда № 22-11-00051, https://rscf.ru/project/22-11-00051/

Для цитирования: *He Y., Petrosyan L. A.* A note on cooperative differential games with pairwise interactions // Вестник Санкт-Петербургского университета. Прикладная математика. Информатика. Процессы управления. 2024. Т. 20. Вып. 1. С. 91–108.
https://doi.org/10.21638/11701/spbu10.2024.108

Предлагается дифференциальная игра с парным взаимодействием. Вершины в сети — это игроки, а ребра — связи между ними. При этом рассматривается кооперативный случай. Вводится новая характеристическая функция и доказывается ее выпуклость. Ядро используется в качестве кооперативного принципа оптимальности. Характеристическая функция позволяет построить устойчивое во времени (динамически устойчивое) решение, такое как вектор Шепли и ядро.

Ключевые слова: кооперативные игры, дифференциальные сетевые игры, парное взаимодействие, характеристическая функция, вектор Шепли, состоятельность по времени.

К о н т а к т н а я и н ф о р м а ц и я:

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