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The method of penalty functions in the analysis of optimal control problems of Navier — Stokes evolutionary systems with a spatial variable in a network-like domain

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The article considers the Navier — Stokes evolutionary differential system used in the mathematical description of the evolutionary processes of transportation of various types of liquids through network or main pipelines. The Navier — Stokes system is considered in Sobolev spaces, the elements of which are functions with carriers on n-dimensional network-like domains. These domains are a totality of a finite number of mutually non-intersecting subdomains connected to each other by parts of the surfaces of their boundaries like a graph (in applications these are the places of branching of pipelines). Two main questions of analysis are discussed: the weak solvability of the initial boundary value problem of the Navier — Stokes system and the optimal control of this system. The main method of research of weak solutions is the semidigitization of the input system by a time variable, that is the reduction of a differential system to a differential-difference system, and using a priori estimates for weak solutions of boundary value problems to prove the theorem of the existence of a solution of the input differential system. For the optimal control problem a minimizing functional (the penalty function) and a family of the approximate functional with parameters that characterize the "penalty" for failure to fulfill the equations of state of the system are introduced. At the same time, a special Hilbert space is created, the elements of which are pairs of functions that describe the state of the system and controlling actions. The convergence of the sequence of such functions to the optimal state of the system and its corresponding optimal control is proved. The latter essentially widen the possibilities of analysis of stationary and nonstationary network-like processes of hydrodynamics and optimal control of these processesd.

Keywords: evolutionary Navier — Stokes system, network-like domain, solvability, optimal control, penalty functions.

1. Introduction. The method of penalty functions is considered, which is enough effectively used in solving the problems of optimization of stationary problems of an applied character [1, 2]. For the analysis of nonstationary problems, this method takes into account information about the equation of state [3, and bibliography there], the basis for the use of which was the need for computing problems. The method of the penalty functions is set out on the example of the problems of optimal starting and distributed control of the

Navier—Stokes system, which are meet in practice, but it is a general method and is used with minor changes in other optimal control problems [4–8]. In the first case, control determines the initial condition of the system, in the second case, control determines the density of external forces of actions on the system; in both cases, the physical problem is to obtain a given vector velocity field at a given final point in time.

2. Designations and concepts. In Euclidean space $\mathbb{R}^n$ $n \geqslant 2$, consider the bounded domain $\Im$, consisting of subdomains $\Im_l$ ($l \in I_N = \{1, 2, ..., N\}$), pairwise connected by M, $1 \leqslant M \leqslant N-1$, the nodal places ω_j ($j \in I_M = \{1, 2, ..., M\}$): $\Im = \hat{\Im} \bigcup \hat{\omega}$, where $\hat{\Im} = \bigcup\limits_{l=1}^{N} \Im_l$, $\hat{\omega} = \bigcup\limits_{j=1}^{M} \omega_j$ and $\Im_l \bigcap \Im_{l'} = \varnothing$ $(l \neq l^{'})$, $\omega_j \bigcap \omega_{j'} = \varnothing$ $(j \neq j^{'})$, $\Im_l \bigcap \omega_j = \varnothing$. Such a domain will be called network-like [9, 10]. The subdomains $\Im_l$ in the nodal places have common boundaries in the form of adjoining surfaces. For fixed $j \in I_M$ the nodal place ω_j is determined by a set of the adjoining subdomains. Namely, each fixed the nodal place ω_j ($j \in I_M$) is adjoined by m_j the domains $\Im_{l_\iota}$, $l_\iota \in I_N(j) = \left\{l_1, l_2, ..., l_{m_j}\right\} \subset I_N$, $\iota = \overline{1, m_j}$, its parts of the boundaries $\partial\Im_{l_\iota}$ which are designated through $S_{j\,\iota} \subset \partial\Im_{l_\iota}$ ($\operatorname{meas} S_{j\,\iota} > 0$), $\iota = \overline{1, m_j}$, in addition $S_j = S_{j\,1} = \bigcup\limits_{\iota=2}^{m_j} S_{j\,\iota}$. Thus, the nodal place ω_j is the branch locus of the domain $\Im$ and is characterized by the surface S_j. The boundary $\partial\Im$ of the domain $\Im$ is defined by the ratio $\partial\Im = \bigcup\limits_{l=1}^{N} \partial\Im_l \backslash \bigcup\limits_{j=1}^{M} S_j$. Everywhere below we consider the adjoining surfaces $S_{j\,s}$ smooth, subdomains $\Im_l$ — star-shaped relative to some ball, its own for each $\Im_l$.

Note that the domain $\Im$ is structured by analogy with the geometric graph-tree [9]. Each subdomain $\Im_l$ at a particular nodal place may be adjoin to one or rather other subdomains, while having one or more adjoining surfaces (for a graph, analogues of nodal places are nodes of conjugation with other edges). Note also that any subdomain of domain $\Im$ can have a network-like structure with its own number of nodal places.

Further, the issues of formation and analysis of a mathematical model of transportation of viscous liquids through complexly structured carriers, which in the applications are different types of pipeline networks, are considered.

For functions $Y(x,t) = \{y_1(x,t), y_2(x,t), ..., y_n(x,t)\}$, $x, t \in \Im_T = \Im \times (0,T)$ ($x = \{x_1, x_2, ..., x_n\}$, $T < \infty$) consider the system

$$\tfrac{\partial Y}{\partial t} - \nu \Delta Y + \sum_{i=1}^{n} Y_i \tfrac{\partial Y}{\partial x_i} = f - \operatorname{grad} p, \tag{1}$$

$$\operatorname{div} Y = 0 \ \left(\sum_{i=1}^{n} \tfrac{\partial Y}{\partial x_i} = 0 \right). \tag{2}$$

Determine the conditions for adjoining the subdomains of the domain $\Im$ by the ratios

$$Y(x,t)|_{x \in S_{j\,\iota} \subset \partial\Im_{l_1}} = Y(x,t)|_{x \in S_{j\,\iota} \subset \partial\Im_{l_\iota}}, \ \ \iota = \overline{2, m_j}, \tag{3}$$

$$\int\limits_{S_j} \tfrac{\partial Y(x,t)}{\partial n_j} ds + \sum_{\iota=2}^{m_j} \int\limits_{S_{j\,\iota}} \tfrac{\partial Y(x,t)}{\partial n_{j\,\iota}} ds = 0, \tag{4}$$

on the surfaces S_j, $S_{j\,\iota}$ ($\iota = \overline{1, m_j}$) of all nodal place ω_j, $j = \overline{1, M}$, and at $t \in (0,T)$. Here vectors n_j and $n_{j\,\iota}$ are external normals to S_j and $S_{j\,\iota}$, respectively, $\iota = \overline{1, m_j}$, $j = \overline{1, M}$. Initial and boundary conditions are determined by the relations

$$Y(x,t)|_{t=0} = Y_0(x), \ \ x \in \Im, \tag{5}$$

$$Y(x,t)|_{x\in\partial\Im} = 0. \tag{6}$$

The relations (1)–(6) define the initial boundary value problem relative to the functions $Y(x,t)$, $p(x,t)$ (hereinafter the differential system (1)–(6)) in a closed network-like domain $\overline{\Im}_T$ ($\overline{\Im}_T = (\Im\bigcup\partial\Im)\times[0,T]$).

In applied questions of mathematical modeling of the processes of transportation of viscous liquids, the network-like domain $\Im$ at $n=3$ belongs to Euclidean space $\mathbb{R}^3$ and models a pipeline network of complex structure or a main line hydraulic system, being a carrier of hydraulic flow (multiphase medium). The function $Y(x,t)$ characterizes the vector of flow speed in $\Im_T$, equations (1), (2) define the Navier — Stokes evolutionary system, which simulates the flow of a liquid with viscosity $\nu>0$ on the carrier $\Im$, the ratios (3), (4) determine the law of flow of fluid flows at the places of branching of the carrier $\Im$, $p(x,t)$ is pressure.

Remark 1. It should be noted that one could use other adjoining conditions, for example,

$$Y|_{S_j^-} = Y|_{S_j^+},\ \sum_{\iota=2}^{m_j}\frac{\partial Y(x)}{\partial n_{j\,\iota}^-}|_{S_{j\,\iota}^-} + \sum_{\iota=2}^{m_j}\frac{Y(x)}{\partial n_{j\,\iota}^+}|_{S_{j\,\iota}^+} = 0,$$

S_j^-, S_j^+ and $S_{j\,\iota}^-$, $S_{j\,\iota}^+$ are one-sided surfaces for S_j, $S_{j\,\iota}$, and $n_{j\,\iota}^-$, $n_{j\,\iota}^+$ are their corresponding normals [11]. The choice of representation of the conditions of adjoining is at the disposal of the researcher and is determined depending on the pursuit purposes. A natural requirement that must be satisfied is the requirement of solvability of the obtained problem, as well as the preservation of the theorem of uniqueness, if the latter corresponds to the spirit of applied research.

3. Solvability of the Navier — Stokes system. The analysis of the solvability of the differential system (1)–(6) is based on the study of the differential-difference system of the form

$$\begin{gathered}\tfrac{1}{\tau}[Y(k)-Y(k-1)] - \nu\Delta Y(k) + \\ +\sum_{i=1}^{n} Y_i(k)\frac{\partial Y(k)}{\partial x_i} = f_\tau(k) - \mathrm{grad}\,p(k),\end{gathered} \tag{7}$$

$$\mathrm{div}\,Y(k) = 0,\ \ k=1,2,...,K,\ \ y(0) = Y_0(x), \tag{8}$$

$$Y(k)|_{x\in\partial\Im} = 0,\ \ k=1,2,...,K, \tag{9}$$

where the following notations are used: $\tau = T/K$ is the step of dividing the segment $[0,T]$ with the dots $k\tau$ ($k=1,2,...,K-1$); $Y(k) := Y(x;k)$, $Y(k)_t := \frac{1}{\tau}[Y(k)-Y(k-1)]$, $f_\tau(k) := f_\tau(x;k) = \frac{1}{\tau}\int\limits_{(k-1)}^{k\tau} f(x,t)dt$ and $p_\tau(k) := p_\tau(x;k) = \frac{1}{\tau}\int\limits_{(k-1)}^{k\tau} p(x,t)dt$ ($k=1,2,...,K$).

Let denote through $L_2(\Im)^n$ the space of the real Lebesgue measurable vector-function $u(x) = \{u_1(x,t), u_2(x,t), ..., u_n(x,t)\}$, $x=(x_1,x_2,...,x_n)\in\mathbb{R}^n$. The scalar product and the norm in $L_2(\Im)^n$ are defined by the equations $(u,v) = \int\limits_{\Im} u(x)v(x)dx$ and $\|u\| = \sqrt{(u,u)}$, respectively (here $\int\limits_{\Im}\phi(x)dx = \sum\limits_{l=1}^{N}\int\limits_{\Im_l}\phi(x)dx$). Next, let $D(\Im)^n$ is the space of infinitely differentiable functions with compact carrier in $\Im$ and $\mathfrak{D}(\Im)^n = \{\phi : \phi\in D(\Im)^n,\ \mathrm{div}\phi = 0\}$. Space $\mathcal{H}(\Im)$ is defined by the closure $\mathfrak{D}(\Im)^n$ in $L_2(\Im)^n$, and space $\mathcal{H}^1(\Im)$ consists of functions $\phi(x)\in\mathcal{H}(\Im)$ having generalized derivatives $\frac{\partial\phi}{\partial x}\in L_2(\Im)^n$. The scalar product and the norm in $\mathcal{H}^1(\Im)$ are defined by the equations $(\mu,\rho)_1 = (\mu,\rho) + \left(\frac{\partial\mu}{\partial x}, \frac{\partial\rho}{\partial x}\right)$ and $\|\phi\|_1 = \left(\|\phi\|^2 + \|\frac{\partial\phi}{\partial x}\|^2\right)^{1/2}$, respectively. To describe the state space of the differential-difference system (7)–(9) we introduce space $V_0^1(\Im)$ as a closure in space $\mathcal{H}^1(\Im)$ of a set of elements $\phi \in \mathfrak{D}(\Im)^n$ satisfying the conditions $\int\limits_{S_j} \frac{\partial\phi(x)}{\partial n_j} ds + \sum\limits_{\iota=2}^{m_j} \int\limits_{S_{j\,\iota}} \frac{\phi(x)}{\partial n_{j\,\iota}} ds = 0$.

The analysis of the differential-difference system (7)–(9) is preceded by consideration of two differential forms $\rho(u,v) = \sum\limits_{i,j=1}^{n} \int\limits_{\Im} \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} dx$, $\tilde\rho(u,v,\omega) = \sum\limits_{i,k=1}^{n} \int\limits_{\Im} u_k \frac{\partial v_i}{\partial x_k}\omega_i dx$, linear for each of their fixed elements u, v and ω. The entered forms are defined on the functions u, v and ω, for which integrals $\int\limits_{\Im} \frac{\partial u_j}{\partial x_i}\frac{\partial v_j}{\partial x_i} dx$ and $\int\limits_{\Im} u_k \frac{\partial v_i}{\partial x_k}\omega_i dx$ converge.

Further discussion will require the following statements (see also [3, p. 88]).

Lemma 1. *The differential form $\rho(u,v)$ is continuous by u, v on $V_0^1(\Im)\times V_0^1(\Im)$, the differential form $\tilde\rho(u,v,\omega)$ is continuous by u, v, ω on $L_4(\Im)^n \times V_0^1(\Im) \times L_4(\Im)^n$.*

Lemma 2. *For arbitrary u, ω of space $V_0^1(\Im)$ there are equalities*:

1) $\tilde\rho(u,u,\omega) = -\tilde\rho(u,\omega,u)$,

2) $\tilde\rho(u,\omega,\omega) = 0$,

3) $\tilde\rho(\omega,\omega,\omega) = 0$.

Lemma 3. *If the sequences $\{u_m\}_{m\geqslant1}$, $\{v_m\}_{m\geqslant1}$ weakly converge in $L_2(\Im)^n$ to u and v, then the sequence $\{u_m v_m\}_{m\geqslant1}$ weakly converges in $L_2(\Im)^n$ to uv.*

The following approach for analyzing the weak solvability of the system (1)–(6) is based on the construction of a priori estimates of the solutions of the differential-difference system (7)–(9) and use of the Galerkin method, which assume look for functions $Y(k) \in V_0^1(\Im)$, $k = 1, 2, ..., K$, in the form of expansions on a special basis of space $V_0^1(\Im)$ — system of generalized eigenfunctions of the operator $\Delta Y = \sum\limits_{i=1}^{n} \frac{\partial^2 Y}{\partial x_i^2}$. Such a system forms the basis in the spaces $V_0^1(\Im)$ and $L_2(\Im)^n$ (proof similar to represented in the work [12, p. 96]).

Remark 2. Can be replaced the boundary condition (6) with a more general $\frac{\partial U}{\partial \mathbf{n}} + \sigma U|_{\partial\Im} = 0$, where the constant σ is her for each subdomain $\Im_l \subset \Im$, $\frac{\partial U}{\partial \mathbf{n}}$ is the derivative of normal $\mathbf{n}$ to the surface $\partial\Im$. The spectral problem in this case is considered in the space $V^1(\Im)$, the elements of which differ from the elements $V_0^1(\Im)$ by the absence for them of the condition of equality to zero on the boundary $\partial\Im$, the integral identity takes the form $\nu\sum\limits_{i=1}^{n}\left(\frac{\partial U}{\partial x_i}, \frac{\partial\eta}{\partial x_i}\right) + \sigma(U,\eta)_{\partial\Im} = \lambda(U,\eta)$ $\forall\eta(x) \in V_0^1(\Im)$, here $(\cdot,\cdot)_{\partial\Im}$ is scalar product on $\partial\Im$. The properties of spectral characteristics remain invariable.

Let us turn to the issue of constructing a priori estimates of the weak solution of the differential-difference system (7)–(9).

Let the initial data of $Y_0(x)$, $f(x,t)$ of the differential system (1)–(6) satisfy the conditions of $Y_0(x) \in V_0^1(\Im)$, $f(x,t) \in L_{2,1}(\Im_T)^n$ (the space $L_{2,1}(\Im_T)^n$ consists of all elements $u \in L_1(\Im_T)^n$ with a finite norm $\|u\|_{2,1} = \int\limits_0^T (\int\limits_{\Im} u(x,t)^2 dx)^{1/2} dt$. The latter means that for the differential-difference system (7)–(9) the original data $Y_0(x)$, $f_\tau(k)$ are the elements of $V_0^1(\Im)$, $L_2(\Im)^n$, respectively.

Definition 1. The set of functions $\{Y(k) \in V_0^1(\Im),\ k = 1, 2, ..., K\}$ for which $Y(k)$ satisfies the ratio

$$(Y(k)_t, \eta) + \nu\rho(Y(k),\eta) + \tilde\rho(Y(k), Y(k), \eta) = (f_\tau(k), \eta), \quad Y(0) = Y_0(x), \tag{10}$$

for fixed k $(k=1,2,...,K-1)$ and arbitrary function $\eta(x)\in V_0^1(\Im)$ is called the weak solution of the differential-difference system (7)–(9).

Taking into account basis property of the set of generalized eigenfunctions $\{U_i(x)\}_{i\geqslant 1}$ in space $V_0^1(\Im)$, to determine the approximations $Y_m(k)=\sum\limits_{i=1}^{m} g_{i,m}^{k}U_i(x)$ of the functions $Y(k)$, $k=1,2,...,K$, of the weak solution of the differential-difference system (7)–(9) consider the system

$$(Y_m(k)_t,U_i)+\nu\rho(Y_m(k),U_i)+\tilde{\rho}(Y_m(k),Y_m(k),U_i)= \\ =(f_\tau(k),U_i),\quad i=1,2,...,m,\quad k=1,2,...,K, \tag{11}$$

$$Y_m(0)=Y_{0\,m}(x), \tag{12}$$

where $Y_{0\,m}(x)=\sum\limits_{i=1}^{m} g_{i\,m}^{0}U_i(x)$ ($g_{i\,m}^{0}$ is const), $Y_{0\,m}(x)\to Y_0(x)$ in norm $\mathcal{H}^1(\Im)$.

First, we get a priori estimates of the norms of functions $Y(k)$, $k=1,2,...,K$, through the norms of the initial data $Y_0(x)$, $f_\tau(k)$.

Theorem 1. *When $Y_0(x)\in V_0^1(\Im)$, $f_\tau(k)\in L_2(\Im)^n$ $(k=1,2,...,K)$ for $Y_m(k)$, $k=1,2,...,K$, of the system* (11) *occur*

$$1)\ \|Y_m(k)\|\leqslant\|Y_m(0)\|+2\|f_\tau(k)\|_{2,1}^{'},$$
$$2)\ \|Y_m(k)\|^2+2\tau\nu\sum_{k^{'}=1}^{k}\|\frac{\partial Y_m(k^{'})}{\partial x}\|^2\leqslant C\left(\|Y_0\|^2+(\|f_\tau(k)\|_{2,1}^{'})^2\right),$$

with a constant C independent of τ, $\|f_\tau(k)\|_{2,1}^{'}=\tau\sum\limits_{k^{'}=1}^{k}\|f_\tau(k^{'})\|$, $k=\overline{1,K}$.

P r o o f. From the ratio $Y(k-1)=Y(k)-\tau Y(k)_t$ follows $2\tau(Y(k),Y(k)_t)=Y^2(k)+\tau^2Y(k)_t^2-Y^2(k-1)$. Multiply the ratios (11), (12) by $2\tau g_{i,m}^{k}$ and sum by i from 1 to m, we get

$$Y_m^2(k)-Y_m^2(k-1)+\tau^2Y_m^2(k)_t+2\tau\nu\rho(Y_m(k),Y_m(k))= \\ =2\tau(f_\tau(k),Y_m(k)),\ \ k=1,2,...,K,$$

taking into account $\rho(Y_m(k),Y_m(k),Y_m(k))=0$ (lemma 2, statement 3), where the inequalities

$$\|Y_m(k)\|^2-\|Y_m(k-1)\|^2+\tau^2\|Y_m(k)_t\|^2+2\tau\nu\|\frac{\partial Y_m(k)}{\partial x}\|^2\leqslant \\ \leqslant 2\tau\|f_\tau(k)\|\|Y_m(k)\|,\ \ k=1,2,...,K, \tag{13}$$

and their obvious consequences

$$\|Y_m(k)\|^2-\|Y_m(k-1)\|^2\leqslant 2\tau\|f_\tau(k)\|\|Y_m(k)\|,\ \ k=1,2,...,K, \tag{14}$$

come from.

Let $\|Y_m(k)\|+\|Y_m(k-1)\|>0$. Taking into account $\frac{\|Y_m(k)\|}{\|Y_m(k)\|+\|Y_m(k-1)\|}\leqslant 1$ and dividing the ratio (14) by $\|Y_m(k)\|+\|Y_m(k-1)\|$, we come to inequalities

$$\|Y_m(k)\|-\|Y_m(k-1)\|\leqslant 2\tau\|f_\tau(k)\|,\quad k=1,2,...,K. \tag{15}$$

If $\|Y_m(k)\|+\|Y_m(k-1)\|=0$, then from the ratio (14) follows $0\leqslant 2\tau\|f_\tau(k)\|\,\|Y_m(k)\|$ and $\|Y_m(k)\|^2-\|Y_m(k-1)\|^2\leqslant 2\tau\|f_\tau(k)\|\,\|Y_m(k)\|$, $k=1,2,...,K$, we come again to (15).

Summing up the inequalities (15) by $k^{'}$ from 1 to k, we finally get the first estimate in the statements of the theorem:

$$\begin{gathered}\|Y_m(k)\| \leqslant \|Y_m(0)\| + 2\sum_{k^{'}=1}^{k} \tau\|f_\tau(^{'})\| = \\ = \|Y_0\| + 2\|f_\tau(k)\|_{2,1}^{'}, \quad k = 1,2,...,K,\end{gathered} \tag{16}$$

where $\|f_\tau(k)\|_{2,1}^{'} = \sum\limits_{k^{'}=1}^{k} \tau\|f_\tau(k^{'})\|$ ($\|\cdot\|_{2,1}^{'}$ is “semi-discrete” analogue norm $\|\cdot\|_{2,1}$ of the space $L_{2,1}(\Im_T)^n$).

Summing up the inequalities (13) by $k^{'}$ from 1 to k and using estimates (16), we come to the second estimate in the statements of the theorem:

$$\begin{gathered}\|Y_m(k)\|^2 + 2\tau\nu\sum_{k^{'}=1}^{k}\|\tfrac{\partial Y_m(k^{'})}{\partial x}\|^2 \leqslant \|Y_m(k)\|^2 + \\ +\tau^2\sum_{k^{'}=1}^{k}\|Y_m(k)_t\|^2 + 2\tau\nu\sum_{k^{'}=1}^{k}\|\tfrac{\partial Y_m(k^{'})}{\partial x}\|^2 \leqslant \\ \leqslant C\left(\|Y_0\|^2 + (\|f_\tau(k)\|_{2,1}^{'})^2\right), \quad k = 1,2,...,K,\end{gathered} \tag{17}$$

where the constant C depends on ν, T and not depend on τ.

Remark 3. The a priori estimates presented by the statements of theorem 1 (see (16), (17)) are the basis for obtaining the conditions for the solvability of the differential system (1)–(6).

Let move on to the analysis of the differential system (1)–(6) and, above all, introduce the necessary functional spaces. Denote through $W^{1,0}(\Im_T)$ a space whose elements $u(x,t)$ together with their generalized derivatives $\frac{\partial u(x,t)}{\partial x}$ belong to $L_2(\Im_T)^n$, $\|u\|_{W^{1,0}(\Im_T)} = \left(\|u\|^2 + \|\frac{\partial u}{\partial x}\|^2\right)^{1/2}$. Let further $W^1(\Im_T)$ is a space whose elements together with their derivatives $\frac{\partial u(x,t)}{\partial x}$, $\frac{\partial u(x,t)}{\partial t}$ belong to $L_2(\Im_T)^n$, $\|u\|_{W^1(\Im_T)} = \left(\|u\|^2 + \|\frac{\partial u}{\partial t}\|^2 + \|\frac{\partial u}{\partial x}\|^2\right)^{1/2}$. The spaces $W^{1,0}(\Im_T)$ and $W^1(\Im_T)$ have the following general properties: 1) their elements are continuous in the norm $L_2(\Im)^n$; 2) traces of their elements on sections $\Im_T$ by planes $t = t_0$ (here t_0 is a arbitrary number of intervals $(0,T)$) are elements of $L_2(\Im)^n$. Next, we introduce two sets $\Omega_1(\Im_T) \subset W^{1,0}(\Im_T)$, $\Omega_2(\Im_T) \subset W^1(\Im_T)$ so that their elements under fixed $t \in (0,T)$ belong to $V_0^1(\Im)$. The closure $\Omega_1(\Im_T)$, $\Omega_2(\Im_T)$ in the corresponding spaces $W^{1,0}(\Im_T)$, $W^1(\Im_T)$ denote by $W_0^{1,0}(\Im_T)$, $W_0^1(\Im_T)$. From what has been said follows $u(x,t)|_{\partial\Im} = 0$, if $u(x,t) \in W_0^{1,0}(\Im_T)$ or $u(x,t) \in W_0^1(\Im_T)$. As above, we take that $Y_0(x) \in V_0^1(\Im)$, $f(x,t) \in L_{2,1}(\Im_T)^n$.

Definition 2. A set

$$\{Y(x,t), p(x,t): \quad Y(x,t) \in W_0^{1,0}(\Im_T),\ p(x,t) \in C(\Im_T)\}$$

is called a weak solution of a differential system (1)–(6), if $Y(x,t)$ satisfies the relation

$$\begin{gathered}-\int\limits_{\Im_T} Y(x,\tau)\frac{\partial\eta(x,\tau)}{\partial\tau}dxd\tau + \nu\int\limits_0^T \rho(Y,\eta)d\tau + \int\limits_0^T \tilde{\rho}(Y,Y,\eta)d\tau = \\ = \int\limits_{\Im} Y_0(x)\eta(x,0)dx + \int\limits_{\Im_T} f(x,\tau)\eta(x,\tau)dxd\tau\end{gathered} \tag{18}$$

for an arbitrary function $\eta(x,t) \in W_0^1(\Im_T)$, $\eta(x,T) = 0$.

Remark 4. By virtue of definition 2 for a function $p(x,t)$ it is necessary that the relation $(\mathbf{grad\,p(x,t)}, \eta(\mathbf{x},\tau)) = \mathbf{0}$ at any $\eta(x,t)$ from $W_0^1(\Im_T)$. The latter is possible, for example, when $p(x,t)$ it belongs to the class $C(\Im_T)$. Note also that in many application problems of continuum transport, the function $p(x,t)$ refers to the input data, therefore its existence not depend on the existence of the function $Y(x,t) \in W_0^{1,0}(\Im_T)$.

Next, the question of the weak solvability of the differential system (1)–(6) is considered [13] (see also [12, p. 189]).

Theorem 2. *The fulfillment of the conditions* $Y_0(x) \in V_0^1(\Im)$, $f(x,t) \in L_{2,1}(\Im_T)^n$ *guarantee a weak solvability of the initial boundary value problem* (1)–(6).

P r o o f. Based on the solution $\{Y(k) \in V_0^1(\Im),\ k = 1,2,...,K\}$ of the differential-difference system (7)–(9) , we introduce a function $Y_K(x,t)$ of the form $Y_K(x,t) = Y(k)$, $t \in ((k-1)\tau,\ k\tau],\ k = 1,2,...,K,\ Y_K(x,0) = Y_0(x)$ (piecewise constant interpolations by a time variable t for $Y(k)$). Belonging $u_K(x,t)$ to space $W_0^{1,0}(\Im_T)$ is obvious. For the function $u_K(x,t)$ the estimates of theorem 1 are valid (inequalities (16) and (17)) and, consequently, the inequality

$$\|Y_K\| + \|\tfrac{\partial Y_K}{\partial x}\| \leqslant C^* \tag{19}$$

is correct for it, a constant $C^* > 0$ independent of τ. A similar representation through $f(x;k)$, $k = 1,2,...,K$, has the function $f_K(x,t)$: $f_K(x,t) = f(x;k)$, $t \in ((k-1)\tau,\ k\tau]$, $k = 1,2,...,K$. Let $K \to \infty$ $(\tau \to 0)$, then it follows from inequality (19) that from the sequence $\{Y_K(x,t)\}$ can be distinguish a subsequence $\{\widetilde{Y}_K(x,t)\}$ that weak converge to the element $Y(x,t) \in W_0^{1,0}(\Im_T)$. Let us show that $Y(x,t)$ is the weak solution of the differential system (1)–(6). To do this, we will establish that $Y(x,t)$ satisfies the identity (18) of definition 2 for any $\eta(x,t) \in C^1(\Im_{T+\tau})^n$, which satisfies the conditions for adjoining (3), (4) under any $t \in (0,T)$ and for which the conditions are met $\eta|_{\partial\Im_T} = 0$, $\eta|_{t\in[T,T+\tau]} = 0$. Functions $\eta(k)$ are defined by $\eta(x,t)$ using the equals $\eta(k) = \eta(x,k\tau)$, $k = 1,2,...,K$, while $\eta(k)_{t'} = \frac{1}{\tau}[\eta(k{+}1){-}\eta(k)]$ (difference relations $\eta(k)_{t'}, \eta(k)_t = \frac{1}{\tau}[\eta(k){-}\eta(k{-}1)]$ are the right and left approximations $\frac{\partial\eta}{\partial t}$ $t = k\tau$, respectively). As for $Y_K(x,t)$, by functions $\eta(k)$ are formed piecewise continuous by the time variable t the approximations $\eta_K(x,t)$, $\frac{\partial\eta_K(x,t)}{\partial x}$ of the functions $\eta(x,t)$, $\frac{\partial\eta(x,t)}{\partial x}$, $\frac{\partial\eta(x,t)}{\partial t}$. Note that $\eta_K(x,t)$, $\frac{\partial\eta_K(x,t)}{\partial x}$, $\frac{\partial\eta_K(x,t)}{\partial t}$ evenly converge on $\overline{\Im}_T$ to $\eta(x,t)$, $\frac{\partial\eta(x,t)}{\partial x}$, $\frac{\partial\eta(x,t)}{\partial t}$ at $K \to \infty$, respectively; $\eta_K(x,t) = 0$, $t \in [T, T+\tau]$.

In the integral identity (14) the function $\eta(x)$ substitute for $\tau\eta(x) = \tau\eta(x;k)$ and sum it on k from 1 to N, we get

$$\begin{aligned}-\tau\sum_{k=1}^{N}\int_{\Im} Y(k)\eta(k)_t dxdt - \int_{\Im} Y_0\eta(k)dx + \nu\sum_{k=1}^{N}\tau\rho(Y(k),\eta(k)) +\\ + \sum_{k=1}^{N}\tau\tilde{\rho}(Y(k),Y(k),\eta(k)) = \sum_{k=1}^{N}\tau\int_{\Im} f_\tau(k)\eta(k)),\end{aligned} \tag{20}$$

taking into account the ratios

$$\tau\sum_{k=1}^{N} Y(k)_t\eta(k) = -\tau\sum_{k=1}^{N} Y(k)\eta(k)_t - Y(0)\eta(k),\ \eta(N) = \eta(N+1) = 0.$$

From the relations (20) it follows directly

$$\begin{aligned}-\int_{\Im_T} Y_K(x,t)\eta_K(x,t)_t dxdt - \int_{\Im} Y_0(x,t)\eta(x,\tau)dx + \nu\int_0^T \rho(Y_K,\eta_K)d +\\ + \int_0^T \tilde{\rho}(Y_K,Y_K,\eta_K)dt = \int_{\Im_T} f_K(x,t)\eta_K(x,t)dxdt.\end{aligned} \tag{21}$$

Passing in (21) to limit by the subsequence $\{Y_K(x,t)\}$ and taking into account the statement of lemma 3, we get the identity (18) of the definition 2 of the weak solution of the differential system (1)–(6). The theorem is proven.

For the vector-function $Y(x,t) = \{y_1(x,t), y_2(x,t), ..., y_n(x,t)\}$, $x,t \in \Im_T$, can consider a linearized Navier—Stokes system, where equation (1) is

$$\frac{\partial Y}{\partial t} - \nu\Delta Y + \mathrm{grad}\, p = f. \tag{22}$$

The systems (22), (2)–(4) and its corresponding initial boundary value problem (22), (2)–(6) in the hydrodynamic theory of transfer processes determines the mathematical model of the laminar flow of a viscous fluid over a network carrier that is described by the domain $\Im_T$.

All the above concepts, definitions and statements are completely preserved, it is necessary only in the ratios (7), (10) and (18) to remove the expression $\sum\limits_{i=1}^{n} Y_i \frac{\partial Y}{\partial x_i}$ and the form $\rho(Y,Y,\eta)$ (statements of lemmas 1–3 for the form $\rho(Y,Y,\eta)$ are not used).

4. The method of penalty functions in the analysis of optimal control problems. Let's denote through $\mathbb{U}$ the Hilbert space of control υ, then $W_0^{1,0}(\Im_T)$ is the space of state $Y(\upsilon)$ of the Navier—Stokes system. In addition $\mathbb{U} = L_2(\Im)^n$ or $\mathbb{U} = L_2(\Im_T)^n$ and therefore $\upsilon := \upsilon(x) \in L_2(\Im)^n$ or $\upsilon := \upsilon(x,t) \in L_2(\Im_T)^n$ for the problem of optimal starting or distributed control, respectively.

Observation of the state $Y(\upsilon)$ of the system is carried out at the final point in time (other types of observations are possible). On a closed convex subset $\mathbb{U}_\partial \subset \mathbb{U}$ the requiring minimization functional

$$J(\upsilon) = \int\limits_{\Im} \left(Y(\upsilon)(x,T) - z_0(x)\right)^2 dx + (N\upsilon, \upsilon)_{\mathbb{U}}, \tag{23}$$

where $z_0(x)$ is given function of space $L_2(\Im)^n$, $N : \mathbb{U} \to \mathbb{U}$ is a linear continuous Hermite operator, $(N\upsilon,\upsilon)_{\mathbb{U}} \geqslant \varsigma\|\upsilon\|^2_{\mathbb{U}}$ ($\varsigma > 0$ is fixed constant).

The problem of optimal starting (distributed) control of the Navier—Stokes system in space $W_0^{1,0}(\Im_T)$ is to find $\inf\limits_{\upsilon\in\mathbb{U}_\partial} J(\upsilon)$, the element $u \in \mathbb{U}_\partial$ is the optimal control of the Navier—Stokes system, which is considered knowingly (a priori) to exist: $\inf\limits_{\upsilon\in\mathbb{U}_\partial} J(\upsilon) = J(u)$.

Let's denote through $\mathbb{Y}$ the set of elements $Y(x,t) \in W_0^{1,0}(\Im_T)$ such that

$$-\int\limits_{\Im} Y_0(x)\eta(x,0)dx - \int\limits_{\Im_T} Y(x,t)\frac{\partial \eta(x,t)}{\partial t}dxdt + \nu\int\limits_0^T \rho(Y,\eta)dt + \int\limits_0^T \tilde{\rho}(Y,Y,\eta)dt =$$
$$= \int\limits_{\Im_T} F(x,t)\eta(x,t)dxdt + \int\limits_{\Im_T} \omega(x,t)\eta(x,t)dxdt, \quad \omega(x,t) \in L_{2,1}(\Im_T),$$

for any $\eta(x,t) \in W_0^1(\Im_T)$, $\eta(x,T) = 0$. For the elements $\mathbb{Y}$ we introduce the norm

$$\|Y\|_{\mathbb{Y}} = \left(\|Y\|^2_{W_0^{1,0}(\Im_T)} + \|\omega\|^2_{L_{2,1}(\Im_T)} + \|Y(\cdot,0)\|^2_{L_2(\Im)}\right)^{1/2},$$

thus

$$\mathbb{Y} = \left\{Y : \ Y \in W_0^{1,0}(\Im_T), \ \omega \in L_{2,1}(\Im_T), \ Y(x,t)|_{t=0} \in L_2(\Im)\right\}.$$

The state of the system (1)–(4) is determined by the initial boundary value problem (1)–(6), moreover in the case of starting control, the control effect $\upsilon(x)$ determines the initial condition (5), i. e. $Y_0(x) := \upsilon(x)$, and in the case of distributed control, the control effect $\upsilon(x,t)$ determines the right side of equation (1): $F(x,t) := \upsilon(x,t)$.

The optimal starting control. The initial condition (5) of the system (1)–(6) is replaced by

$$Y(x,t)|_{t=0} = \upsilon(x), \quad x \in \Im, \tag{24}$$

thus, the state $Y(\upsilon)$ of the Navier — Stokes system is characterized by the initial boundary value problem (1)–(4), (24), (6) and $\mathbb{U} = L_2(\Im)^n$.

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \varepsilon_q) > 0$, $q = 1, 2$, and granting (23) define on $\mathbb{Y} \times \mathbb{U}$ the functional

$$\begin{aligned} J_\varepsilon(Y,\ \upsilon) = \int_\Im (Y(x,T) - z_0(x))^2\, dx + (N\upsilon, \upsilon)_{\mathbb{U}} + \\ + \tfrac{1}{\varepsilon_1} \|\omega - F\|^2_{L_{2,\,1}(\Im_T)} + \tfrac{1}{\varepsilon_2} \int_\Im (Y(x\,,0) - \upsilon(x))^2\, dx, \end{aligned} \tag{25}$$

named in the literature penalty function [3, p. 395]. Multipliers $1/\varepsilon_1,\ 1/\varepsilon_2$ characterize fines if the ratios (1) and (5) are not satisfied.

Consider an auxiliary problem with a parameter $\varepsilon = (\varepsilon_1,\ \varepsilon_2)$ (family of problems) of finding $\inf\limits_{Y\in\mathbb{Y},\ \upsilon\in\mathbb{U}_\partial} J_\varepsilon(Y,\ \upsilon)$ on $\mathbb{Y} \times \mathbb{U}$, approximating the search problem $\inf\limits_{\upsilon\in\mathbb{U}_\partial} J(\upsilon)$ and assume that there exists a pair $\{Y_\varepsilon,\ \upsilon_\varepsilon\}$ for which $\inf\limits_{Y\in\mathbb{Y},\ \upsilon\in\mathbb{U}_\partial} J_\varepsilon(Y,\ \upsilon) = J^\circ_\varepsilon$.

Theorem 3. *Under the assumption that the solution $\{Y_\varepsilon,\ u_\varepsilon\}$ to the problem of finding $\inf\limits_{Y\in\mathbb{Y},\ \upsilon\in\mathbb{U}_\partial} J_\varepsilon(Y,\ \upsilon)$ does exist, takes place*

$$J^\circ_\varepsilon \to J(u), \tag{26}$$

$$\upsilon_\varepsilon \to u \ \ \textit{in the norm space } \mathbb{U}, \tag{27}$$

$$Y_\varepsilon \to Y(u) \ \ \textit{in the norm } \ \|\cdot\|_{\mathbb{Y}} \tag{28}$$

at $\varepsilon = (\varepsilon_1,\ \varepsilon_2) \to 0$.

P r o o f. As mentioned above, the element $u \in \mathbb{U}_\partial$ is the optimal control of the Navier — Stokes system, that means $\inf\limits_{\upsilon\in\mathbb{U}_\partial} J(\upsilon) = J(u) = J^\circ$ and for the state $Y(u)$ (solving the initial boundary value problem (1)–(4), (24) and (6) at $\upsilon(x) = u(x)$) the relations (1), (24) and (6) are satisfied. From the latter and the representation (25) of the functional $J_\varepsilon(Y, \upsilon)$ follows inequality

$$J_\varepsilon(Y_\varepsilon,\ u_\varepsilon) \leqslant J_\varepsilon(Y(u),\ u) = J(u) = J^\circ. \tag{29}$$

From the estimate (29) follows the boundedness $J_\varepsilon(Y_\varepsilon,\ u_\varepsilon)$ for the arbitrary $\varepsilon = (\varepsilon_1,\ \varepsilon_2)$ and, using the expression (25), we come to inequalities

$$J_\varepsilon(Y_\varepsilon,\ u_\varepsilon) \geqslant \varsigma\|u_\varepsilon\|^2_{\mathbb{U}},$$

$$J_\varepsilon(Y_\varepsilon,\ u_\varepsilon) \geqslant \tfrac{1}{\varepsilon_1} \|\omega_\varepsilon - F\|^2_{L_{2,\,1}(\Im_T)} + \tfrac{1}{\varepsilon_2} \int_\Im (Y_\varepsilon(x\,,0) - u_\varepsilon(x))^2\, dx,$$

of which follows

$$\|u_\varepsilon\|_{\mathbb{U}} \leqslant C, \tag{30}$$

$$\|\omega_\varepsilon - F\|_{L_{2,\,1}(\Im_T)} \leqslant C\sqrt{\varepsilon_1}, \tag{31}$$

$$\|Y_\varepsilon(\cdot\,,0) - u_\varepsilon\|_{L_2(\Im)} \leqslant C\sqrt{\varepsilon_2}, \tag{32}$$

where the constant C depends on the value J°. From the ratios (30)–(32) it follows that with $\varepsilon \to 0$ a set of functions $u_\varepsilon(x)$ bounded in $\mathbb{U} = L_2(\Im)^n$, a set of functions $\omega_\varepsilon(x,t) - F(x,t)$ is bounded in $L_{2,1}(\Im_T)^n$, a set of functions $Y_\varepsilon(x\,,0)$ is bounded in $L_2(\Im)^n$, it means a set of functions $Y_\varepsilon(x\,,t)$ is bounded in $W_0^{1,0}(\Im_T)$ and $\mathbb{Y}$. It follows that from a sequence $\{Y_\varepsilon(x\,,t),\ u_\varepsilon(x)\}$ can be extracted a subsequence (let's leave the same notation $\{Y_\varepsilon(x\,,t),\ u_\varepsilon(x)\}$ for it) for which $Y_\varepsilon(x\,,t) \to \tilde{Y}$ weakly converges in $W_0^{1,0}(\Im_T)$ and $u_\varepsilon(x) \to \tilde{u}$ weakly converges in $\mathbb{U} = L_2(\Im)^n$ ($\tilde{u} \in \mathbb{U}_\partial$). Thus, passage to the limit by subsequence leads to the relations $\tilde{\omega}(x,t) - F(x,t) = 0$, $\tilde{Y}(\cdot\,,0) = \tilde{u}$, which means $\tilde{Y}(x,t) = \tilde{Y}(x,t;\,\tilde{u})$.

From the ratio (25) follows

$$J_\varepsilon(Y_\varepsilon, \upsilon_\varepsilon) \geqslant \int\limits_{\Im} \left(Y_\varepsilon(x,T) - z_0(x)\right)^2 dx + (N\upsilon_\varepsilon, \upsilon_\varepsilon)_{\mathbb{U}}, \tag{33}$$

moreover $Y_\varepsilon(x,T)$ weakly converges in $L_2(\Im)^n$ to $\tilde{Y}(x,T)$, and then from inequality (33) follows inequality $\underline{\lim}\, J_\varepsilon(Y_\varepsilon, \upsilon_\varepsilon) \geqslant \int\limits_{\Im} \left(\tilde{Y}(x,T) - z_0(x)\right)^2 dx + (N\tilde{u}, \tilde{u})_{\mathbb{U}}$ or $\underline{\lim}\, J_\varepsilon(Y_\varepsilon, \upsilon_\varepsilon) \geqslant J(\tilde{u})$. The latter, together with the relation (29) means that $\tilde{u} = u$ and the correctness of the statement (26) of the theorem, hence the statements (27), (28), is valid in the sense of weak convergence.

Let us show the validity of the statements (27), (28) in the relevant norms, that is, in the sense of strong convergence. Let's present the functional $J_\varepsilon(Y_\varepsilon, \upsilon_\varepsilon)$ in the form

$$J_\varepsilon(Y_\varepsilon, \upsilon_\varepsilon) = \theta_\varepsilon + \vartheta_\varepsilon - 2\int\limits_{\Im} Y_\varepsilon(x,T)\, z_0(x) dx + \int\limits_{\Im} z_0^2(x) dx,$$

here $\theta_\varepsilon = \int_\Im Y_\varepsilon^2(x,T)\,dx + (Nu_\varepsilon, u_\varepsilon)_{\mathbb{U}}$, $\vartheta_\varepsilon = \frac{1}{\varepsilon_1}\left\|\omega_\varepsilon - F\right\|^2_{L_{2,1}(\Im_T)} + \frac{1}{\varepsilon_2}\left\|Y_\varepsilon(\cdot,\ 0) - u_\varepsilon\right\|^2_{L_2(\Im)}$.

By virtue of

$$J_\varepsilon(Y_\varepsilon, \upsilon_\varepsilon) \to J^\circ = J(u) = \theta - 2\int\limits_{\Im} Y(x,T)\, z_0(x) dx + \int\limits_{\Im} z_0^2(x) dx,$$

where $\theta = \int_\Im Y^2(x,T;u)\,dx + (Nu,u)_{\mathbb{U}}$, it should be $\theta_\varepsilon + \vartheta_\varepsilon \to \theta$. Hence, given $\underline{\lim}\,\theta_\varepsilon \geqslant \theta$, we get $\vartheta_\varepsilon \to 0$, what means $\upsilon_\varepsilon \to u$ in the norm of space $\mathbb{U}$, and $\theta_\varepsilon \to \theta$, which means $Y_\varepsilon \to Y(u)$ in the norm $\|\cdot\|_{\mathbb{Y}}$: the validity of the statements (27), (28) is established, the theorem is proved.

Remark 5. From the reasoning it follows that the estimates (30)–(32) are valid for an arbitrarily small constant C and for sufficiently small ε_1, ε_2.

The optimal distributed control. The method of penalty functions for the analysis of the problem of optimal distributed control of the Navier—Stokes system (1)–(4) it remains invariable, the equation (1), the control space $\mathbb{U}$, the space is slightly changed and functionality $J_\varepsilon(Y,\ \upsilon)$. The equation (1), the control space, and the space are slightly modified. Namely, equation (1) is replaced by

$$\frac{\partial Y}{\partial t} - \nu\Delta Y + \sum_{i=1}^{n} Y_i \frac{\partial Y}{\partial x_i} + \operatorname{grad} p = \upsilon(x,t), \tag{34}$$

$\mathbb{U} = L_2(\Im_T)^n$, $\upsilon(x,t) \in \mathbb{U}$, functional $J_\varepsilon(Y,\ \upsilon)$ take the form

$$J_\varepsilon(Y,\ \upsilon) = \int\limits_{\Im} \left(Y(x,T) - z_0(x)\right)^2 dx + (N\upsilon, \upsilon)_{\mathbb{U}} +$$
$$+ \frac{1}{\varepsilon_1}\left\|\omega - \upsilon\right\|^2_{L_{2,1}(\Im_T)} + \frac{1}{\varepsilon_2}\int_\Im \left(Y(x\,,0) - \upsilon(x)\right)^2 dx,$$

the state $Y(v) \in W_0^{1,0}(\Im_T)$ of the Navier — Stokes system is determined by the systems (5), (34). Further reasoning almost verbatim repeats the above.

5. The method of penalty functions for the linearized Navier — Stokes system. In the previous case, when the nonlinear Navier — Stokes system was considered, the analysis of optimal control was limited to obtaining the necessary conditions for a minimum of functional (the penalty function) under the supposition of the existence of optimal control. The case of the linearized Navier — Stokes system by favorable to the obtaining of the necessary and sufficient conditions, since additional properties of the linearized system are used along this way, it is possible to prove the existence of a unique optimal control. We show this on the example of starting control.

For a linearized Navier — Stokes system (2)–(4), (22) the state of which is defined as a weak solution in the space $W_0^{1,0}(\Im_T)$ of the system

$$\frac{\partial Y}{\partial t} - \nu \Delta Y + \operatorname{grad} p = F(x,t) \tag{35}$$

with the condition (24) ($v(x) \in \mathbb{U} = L_2(\Im)^n$ is control effect), the problem of optimal starting control is considered.

For this case, the statements of Section 3 remain valid, with the only difference that they do not contain the expression $\sum\limits_{i=1}^{n} Y_i \frac{\partial Y}{\partial x_i}$ and its form $\tilde{\rho}(Y,\, Y,\, \eta)$. The introduced above functional $J(v)$, auxiliary space $\mathbb{Y}$ and functional $J_\varepsilon(Y,\, v)$ are also preserved, where the expression $\sum\limits_{i=1}^{n} Y_i \frac{\partial Y}{\partial x_i}$ and its form $\tilde{\rho}(Y,\, Y,\, \eta)$ are also absent. A essential difference from the previous consideration is the possibility to establish the uniqueness of the solution of the optimal starting control problem (problem $\inf\limits_{v \in \mathbb{U}_\partial} J(v)$) and the uniqueness of the auxiliary problem with the parameter $\varepsilon = (\varepsilon_1,\ \varepsilon_2)$ search for $\inf\limits_{Y \in \mathbb{Y},\ v \in \mathbb{U}_\partial} J_\varepsilon(Y,\, v)$, approximating problem $\inf\limits_{v \in \mathbb{U}_\partial} J(v)$.

The uniqueness of the solution of the problem $\inf\limits_{v \in \mathbb{U}_\partial} J(v)$ is a consequence of the following statements, similar to those proven in the work [14].

Theorem 4. *The operator of the transition from control $v(x) \in \mathbb{U} = L_2(\Im)^n$ to $Y(v) \in W_0^{1,0}(\Im_T)$ continuous.*

Theorem 5. *The problem of optimal starting control has a unique solution.*

The proof of the statement of theorem 4 uses the linearity of the operator of the problem (35), (24) and the a priori estimates given in theorem 1. The statement of theorem 2 is based on the property of coercivity of the homogeneous part of the second degree of the quadratic form of the functional $J_\varepsilon(Y,\ v)$ and the statement of theorem 4.

The uniqueness of the auxiliary problem $\inf\limits_{Y \in \mathbb{Y},\, v \in \mathbb{U}_\partial} J_\varepsilon(Y,\, v)$ is established by the following statement.

Theorem 6. *The problem* $\inf\limits_{Y \in \mathbb{Y},\, v \in \mathbb{U}_\partial} J_\varepsilon(Y,\, v)$ *has a unique solution.*

P r o o f. For the functional $J_\varepsilon(Y,\ v)$ $\left(\text{absent the expression } \sum\limits_{i=1}^{n} Y_i \frac{\partial Y}{\partial x_i}\right)$, consider the part containing the second degrees:

$$q_\varepsilon(Y,\ v) = \int\limits_{\Im} Y^2(x,T)dx + (Nv, v)_{\mathbb{U}} + \frac{1}{\varepsilon_1} \|\omega - F\|^2_{L_{2,\,1}(\Im_T)} + \frac{1}{\varepsilon_2} \int\limits_{\Im} Y^2(x,\, 0)dx.$$

Note that

$$q_\varepsilon(Y,\ v) \geqslant C\left(\|Y\|^2_{\mathbb{Y}} + \|v\|^2_{\mathbb{U}}\right). \tag{36}$$

Indeed, given the inequality $(Nv, v)_{\mathbb{U}} \geqslant \varsigma\|v\|^2_{\mathbb{U}}$ $(\varsigma > 0)$, we come to inequality

$$q_\varepsilon(Y,\ v) \geqslant (\varsigma + \tfrac{1}{\varepsilon_2})\,\|v\|^2_{\mathbb{U}} + \int\limits_{\Im} Y^2(x,T)dx + \tfrac{1}{\varepsilon_2}\int\limits_{\Im} Y^2(x,\,0)dx + \\ + \tfrac{1}{\varepsilon_1}\|\omega\|^2_{L_{2,\,1}(\Im_T)} - \tfrac{2}{\varepsilon_2}\,\|v\|_{\mathbb{U}}\,\|Y(\cdot\,,\,0)\|_{L_{2,}(\Im)}\,,$$

from which we get inequality (36) with a constant C, dependent on $\varepsilon_1,\ \varepsilon_2$. The proof complete with the statement of the theorem 1.1 [15, p. 13].

Repeating the reasoning given in the proof of theorem 3, we come to the conclusion: 1) there is a unique solution to the problem of optimal control; 2) a necessary and sufficient condition for the existence of optimal control is the presence of a sequence of pairs $\{Y_\varepsilon,\ v_\varepsilon\}$, for which with each sufficiently small $\varepsilon = (\varepsilon_1,\ \varepsilon_2)$ pair $\{Y_\varepsilon,\ v_\varepsilon\}$ it realizes $\inf\limits_{Y\in\mathbb{Y},\ v\in\mathbb{U}_\partial} J_\varepsilon(Y,\ v)$. This sequence contains a subsequence that weak converge to the optimal pair $\{Y(x\,,t),\ u(x)\}$ (the solving of the problem of finding $\inf\limits_{v\in\mathbb{U}_\partial} J(v)$).

6. Conclusion. The approach presented in the paper explain on the example of the problems of optimal control of the Navier — Stokes evolutionary system with a spatial variable changing in a network-like domain. The penalty function method used in this case is a fairly general method. It can also be used (with minor modifications) to analyze the optimal control problems of stationary Navier — Stokes systems (linear and linearized). The effectiveness of this method essentially increase in connection with the needs of computing tasks of applied nature [16–18]. Note at the same time that the method of the penalty functions can be effectively applied to the numerical solution of the optimization problem in various areas of natural science (see, for example, work [19]).

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Метод штрафных функций в анализе задач оптимального управления эволюционными системами Навье — Стокса с пространственной переменной в сетеподобной области

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Изучается эволюционная дифференциальная система Навье — Стокса, используемая при математическом описании эволюционных процессов транспортировки разного типа жидкостей по сетевым или магистральным трубопроводам. Система Навье — Стокса рассматривается в пространствах Соболева, элементы которых — функции с носителями на n-мерных сетеподобных областях. Эти области есть совокупность конечного числа взаимно не пересекающихся подобластей, соединенных друг с другом частями поверхностей своих границ по типу графа (в приложениях: местах ветвления трубопроводов). Обсуждаются два основных вопроса анализа: слабая разрешимость начально-краевой задачи для системы Навье — Стокса и оптимальное управление этой системой. Основными методами исследования слабой разрешимости являются полудискретизация исходной системы по временной переменной, т. е. редукция дифференциальной системы к дифференциально-разностной, и использование априорных оценок для слабых решений краевых задач при доказательстве теоремы существования решения исходной дифференциальной системы. Для задачи оптимального управления вводятся минимизирующий функционал (функция штрафа) и аппроксимирующее его семейство вспомогательных функционалов с параметрами, которые характеризуют штраф за невыполнение уравнений состояния системы. При этом вводится специальное гильбертово пространство, элементами которого являются пары функций, описывающих состояние системы и управляющие воздействия. Доказывается сходимость последовательности таких функций к оптимальному состоянию системы и ему соответствующему оптимальному управлению. Последнее существенно расширяет возможности анализа стационарных и нестационарных сетеподобных процессов гидродинамики и оптимального управления ими.

Ключевые слова: эволюционная система Навье — Стокса, сетеподобная область, разрешимость, оптимальное управление, штрафные функции.

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