UDC 517.988.6, 517.988.8
MSC 47H09, 47H10

Common fixed point results: New developments on commuting mappings and application in dynamic programming

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For citation: Touail Y. Common fixed point results: New developments on commuting mappings and application in dynamic programming. *Vestnik of Saint Petersburg University. Applied Mathematics. Computer Science. Control Processes*, 2024, vol. 20, iss. 3, pp. 366–375.
https://doi.org/10.21638/spbu10.2024.305

Based on a class of semicontinuous functions, we prove a common fixed point theorem for a pair of commuting mappings. As a consequence, we give another common fixed point for the so-called weakly contractive mappings of type E_T. The proven results are established in the setting of bounded metric spaces without using neither the compactness nor the uniform convexity. Some examples are built to demonstrate the superiority of the obtained results compared to the existing ones in the literature. Furthermore, an application to a system of functional equations arising in dynamic programming is given.

Keywords: common fixed point, weakly contractive maps of type E_T, commuting maps, compactness, uniform convexity.

1. Introduction. The metric fixed point theory goes back to the first decades of the 20$^{\text{th}}$ century, when in 1922 there occurred the famous Banach contraction principle (BCP for short). This result has been used extensively in the study of solutions for differential equations, dynamical systems, models in economy and related areas, game theory, physics, engineering, computer science and other. This principle was the subject of several generalizations by many mathematicians. In this regard it is worth mentioning the important papers by Nemytzki appeared in 1930 [1] and Edelstein published in 1962 [2] where the authors prove that the strict contraction ($d(Tx, Ty) < d(x, y)$, for all $x \neq y \in X$, where X is a metric space) has a unique fixed point if the space is assumed compact. As a generalization of this type of contractions, we refer to a more large class called nonexpansive mappings ($\|Tx - Ty\| \leqslant \|x - y\|$ for all $x, y \in X$, where $(X, \|\cdot\|)$ is Banach space). It is well-known that this type need not have a fixed point in a general Banach space. However, by enriching the space with some geometric properties like uniform convexity, it is possible to have fixed points. In 1965, Browder [3], Göhde [4] and Kirk [5] independently showed one of the most interesting extensions of BCP by proving that every nonexpansive mapping of a closed convex and bounded subset of the Banach space X has a fixed point, if the subset is supposed to be uniformly convex (for each $0 < \varepsilon \leqslant 2$, there exists $\delta > 0$ such that for all $\|x\| \leqslant 1, \|y\| \leqslant 1$) the condition $\|x - y\| \geqslant \varepsilon$ implies that $\|\frac{x+y}{2}\| \leqslant 1 - \delta$ (see [6]).

On the other hand, the idea of a common fixed point of commuting mappings verifying certain contractive conditions in the setting of metric spaces was initiated by Jungck [7] as an extension of BCP in 1976.

At the same time, other mathematicians tried successfully to extend fixed point theory and common fixed point theory to various abstract spaces. One of the most popular is the partial metric spaces introduced by Matthews [8] (1994) as an extension of the concept of the metric space such that the separation axiom $d(x, x) = 0$ of metric's definition is replaced by the condition $\sigma(x, x) \leqslant \sigma(x, y)$, that is, $\sigma(x, x) > 0$ for some x. It is widely recognized that partial metric spaces play an important role in constructing models in the theory of computation (see [9, 10]).

Matthews [8] affirmed that for any partial metric σ on a nonempty set X, there exists an induced metric $d_\sigma : X \times X \to \mathbb{R}^+$ defined by

$$d_\sigma(x, y) = 2\sigma(x, y) - \sigma(x, x) - \sigma(y, y),$$

symmetrically

$$\sigma(x, y) = \alpha(x) + d(x, y) + \alpha(y). \tag{1}$$

In (1) $\alpha : X \to \mathbb{R}^+$ is defined by $\alpha(x) = \frac{1}{2}\sigma(x, x)$ and $d(x, y) = \frac{1}{2}d_\sigma(x, y)$.

We point that the existence of a common fixed point result for commuting mappings f, g on a complete partial metric space X remains valid under the following assumptions (see [8]):

$$\sigma(fx, gy) \leqslant k\sigma(x, y), \tag{2}$$

with $k < 1$ for all $x, y \in X$.

It seems from (2) that the conditions

$$\alpha(gx) \leqslant k\alpha(x), \tag{3}$$

$$\alpha(fx) \leqslant k\alpha(x), \tag{4}$$

for all $x \in X$ are used.

As an improvement of the study in [11], in this paper, we extend conditions (3) and (4) to

$$\alpha(gx) \leqslant \alpha(x), \tag{5}$$

$$\alpha(fx) \leqslant \alpha(x). \tag{6}$$

Very recently in 2021, the authors in [12] proved a common fixed point for commuting mappings f and g satisfying

$$\inf_{x \neq y \in X} \{d(x, y) - d(fx, gy)\} > 0. \tag{7}$$

In a comparison with the famous result proved by the author [2], the existence of a common fixed point for the class mentioned in (7) is established without adding the compactness property to the metric space. (The reader can see [11–17] and references therein, for recent works in this direction.)

In the present work, by weakening conditions (3) and (4) and considering the class of certain lower semicontinuous function α we introduce a new category of mappings

$$\inf_{x \neq y \in X} \{\alpha(x) + d(x, y) + \alpha(y) - \alpha(fx) - d(fx, gy) - \alpha(gy)\} > 0,$$

as a combination with (5)–(7).

For such kind of mappings, a new common fixed point theorem is also presented. In this way, we obtain a very large class compared with other results in the literature. In other words, we prove a common fixed point theorem for a new type of nonexpansive mappings (i.e., $d(gx, gy) \leqslant d(x, y)$) without using neither the compactness nor the uniform convexity which is difficult in the practice. Moreover, motivated by [12, 13, 18], we prove via the first result a common fixed point for the so-called weakly contractive mappings of type E_T.

As an application, in the last section, we show how to employ our main result to solve some problems which appear in dynamic programming. This theory was initiated by Bellman [19] and it is strongly connected with the multistage decision processes [20], in which some functional equations of type

$$f_i(x) = \sup_{y \in D}\{g(x, y) + G_i(x, y, f_i(\rho(x, y)))\} \tag{8}$$

arise. We point that the existence of a solution for (8) is proven under new weak conditions.

2. Preliminaries. The aim of this section is to present some notions and results used in the paper. Throughout the article, we denote by $\mathbb{R}$ the set of all real numbers and by $\mathbb{N}$ the set of all positive integers.

Let (X, τ) be a topological space and $p : X \times X \to [0, \infty)$ be a function. For any $\varepsilon > 0$ and any $x \in X$, let $B_p(x, \varepsilon) = \{y \in X : p(x, y) < \varepsilon\}$.

Definition 1 (Definition 2.1 [21]). The function p is said to be τ-distance if for each $x \in X$ and any neighborhood V of x, there exists $\varepsilon > 0$ such that $B_p(x, \varepsilon) \subset V$.

Definition 2 ([21]). A sequence $\{x_n\}$ in a Hausdorff topological space X is a p-Cauchy if it satisfies the usual metric condition with respect to p (i. e., $\lim_{n,m\to\infty} p(x_n, x_m) = 0$).

Definition 3 (Definition 3.1 [21]). Let (X, τ) be a topological space with a τ-distance p.

• X is S-complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim p(x, x_n) = 0$.

• X is p-Cauchy complete if for every p-Cauchy sequence (x_n), there exists x in X with $\lim x_n = x$ with respect to τ.

• X is said to be p-bounded if $\sup\{p(x, y) : x, y \in X\} < \infty$.

Lemma 1 (Lemma 3.1 [21]). *Let (X, τ) be a Hausdorff topological space with a τ-distance p, then*

• *$p(x, y) = 0$ implies $x = y$;*

• *if (x_n) is a sequence in X such that $\lim_{n\to\infty} p(x, x_n) = 0$ and $\lim_{n\to\infty} p(y, x_n) = 0$, then $x = y$.*

Definition 4 ([8]). A partial metric on a nonempty set X is a function $\sigma : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$:

a) $\sigma(x, x) = \sigma(x, y) = \sigma(y, y)$ if and only if $x = y$;

b) $\sigma(x, x) \leqslant \sigma(x, y)$;

c) $\sigma(x, y) = \sigma(y, x)$;

d) $\sigma(x, z) + \sigma(y, y) \leqslant \sigma(x, y) + \sigma(y, z)$.

The pair (X, σ) is called a partial metric space.

Definition 5 ([8]). Let (X, σ) be a partial metric space. Then

i) a sequence $\{x_n\}$ in X converges to a point $x \in X$ if $\lim_{n\to\infty} \sigma(x, x_n) = \sigma(x, x)$;

ii) a sequence $\{x_n\} \subset X$ is Cauchy if $\lim_{m,n\to\infty} \sigma(x_m, x_n)$ exists and is finite;

iii) X is complete if every Cauchy sequence $\{x_n\} \subset X$ converges to a point $x \in X$, that is, $\lim_{m,n\to\infty} \sigma(x_m, x_n) = \sigma(x, x) = \lim_{n\to\infty} \sigma(x, x_n)$.

Theorem 1 (Theorem 2.1 [12]). *Let (X, τ) be a p-bounded Hausdorff topological space with a τ-distance p. Let f and g be two selfmappings of X, satisfying the conditions:*

• *$f \circ g = g \circ f$;*

• *$p(fx, gy) \leqslant kp(x, y)$, for all $x, y \in X$ and $k < 1$.*

If (X, τ) is a S-complete space, then f and g have a unique common fixed point.

Theorem 2 (Theorem 2.5 [12]). *Let (X, d) be a bounded complete metric space (X, d). Let f and g be two selfmappings of X, satisfying the conditions:*

- $f \circ g = g \circ f$;
- $\inf\limits_{x \neq y} \{d(x, y) - d(fx, gy)\} > 0.$

Then f and g have a unique common fixed point.

Definition 6 (Definition 2.10 [12]). Let (X, d) be a metric space and $f, g : X \longrightarrow X$ be a two selfmappings of X such that $f \circ g = g \circ f$. The mappings f and g are said to be E_T-weakly contractive if $d(fx, gy) \leqslant d(x, y) - \varphi(d(x, y) + 1)$ for all $x, y \in X$, where $\varphi : [1, \infty) \to [0, \infty)$, $\varphi(1) = 0$ and $\inf\limits_{t>1} \varphi(t) > 0$.

Theorem 3 (Theorem 2.11 [12]). *Let (X, d) be a bounded complete metric space and f, g be two E_T-weakly maps on X. Then f and g have a unique common fixed point.*

Lemma 2 (Lemma 3.2 [12]). *Let A be a subset of $\mathbb{R}$ such that* $\sup(A) < +\infty$ *and $G : \mathbb{R} \to \mathbb{R}$ be nondecreasing and continuous function. Then we have the formula*

$$G(\sup A) = \sup G(A).$$

Definition 7. Let (X, d) be a metric space, a function $\alpha : X \to [0, \infty)$ is said to be lower semicontinuous if for all $y \in X$ and $\{x_n\} \subset X$ such that $\lim_{n\to\infty} x_n = y$, we get formula

$$\alpha(y) \leqslant \lim_{n\to\infty} \inf \alpha(x_n).$$

Lemma 3 (Lemma 2.2 [11]). *Let (X, d) be a metric space and $p : X \times X \to \mathbb{R}^+$ be a function defined by*

$$p(x, y) = e^{\alpha(x)+d(x,y)+\alpha(y)} - 1,$$

where $\alpha : X \to \mathbb{R}^+$ is a function. Then p is a τ_d-distance on X where τ_d is the metric topology.

Lemma 4 (Lemma 2.3 [11]). *Let (X, d) be a bounded metric space and $\alpha : X \to \mathbb{R}^+$ is a bounded function. Then the function p defined in Lemma is a bounded τ-distance.*

Lemma 5 (Lemma 2.4 [11]). *Let (X, d) be a complete metric space and $\alpha : X \to \mathbb{R}^+$ is a lower semicontinuous function. Then the function p defined in Lemma is a S-complete τ-distance.*

3. Main results. Now, we are able to prove our main results.

Theorem 4. *Let (X, d) be a bounded complete metric space (X, d). Let f and g be two selfmapping of X satisfying the following conditions:*

- $f \circ g = g \circ f$;
- $\inf\limits_{x \neq y} \{\alpha(x) + d(x, y) + \alpha(y) - \alpha(fx) - d(fx, gy) - \alpha(gy)\} > 0,$

where $\alpha : X \to [0, \infty)$ is a bounded and lower semicontinuous function. Then f and g have a unique common fixed point.

P r o o f. We put $\gamma = \inf\limits_{x \neq y} \{\alpha(x) + d(x, y) + \alpha(y) - \alpha(fx) - d(fx, gy) - \alpha(gy)\}$, and hence

$$\alpha(fx) + d(fx, gy) + \alpha(gy) \leqslant \alpha(x) + d(x, y) + \alpha(y) - \gamma,$$

for all $x \neq y \in X$. Therefore,

$$e^{\alpha(fx)+d(fx,gy)+\alpha(gy)} \leqslant ke^{\alpha(x)+d(x,y)+\alpha(y)},$$

with $k = e^{-\gamma} < 1$. Then
$$p(fx, gy) \leqslant kp(x, y),$$
for all $x, y \in X$, with $p(x, y) = e^{\alpha(x)+d(x,y)+\alpha(y)} - 1$ is the τ-distance defined in Lemma 3. Now, we deduce from Theorem 1, Lemmas 2–5 that f and g have a unique common fixed point. □

Corollary 1. Let (X, d) be a bounded complete metric space (X, d). Let f be a selfmapping of X. If there exists $i, j \in \mathbb{N}$ such that

$$\inf_{x\neq y}\{\alpha(x) + d(x, y) + \alpha(y) - \alpha(f^i x) - d(f^i x, f^j y) - \alpha(f^j y)\} > 0.$$

Then f has a unique fixed point.

P r o o f. We have for all $m, n \in \mathbb{N}$, $f^n \circ f^m = f^m \circ f^n$. Then, f^i and f^j satisfy all conditions of Theorem 4, so there exists a unique $u \in X$ such that $f^i u = f^j u = u$. Also, $f \circ f^i u = f^i \circ fu = fu$ and $f \circ f^j u = f^j \circ fu = fu$, by uniqueness we obtain $fu = u$. □

Corollary 2 (Theorem 2.5 [11])**.** Let (X, d) be a bounded complete metric space (X, d). Let f be a selfmapping of X satisfying the condition

$$\inf_{x\neq y}\{\alpha(x) + d(x, y) + \alpha(y) - \alpha(fx) - d(fx, fy) - \alpha(fy)\} > 0,$$

where $\alpha : X \to [0, \infty)$ is a bounded and lower semicontinuous function. Then f has a unique fixed point.

Example 1. We endow the set $X = \{0, 1, 2\}^2$ with the metric

$$d\big((x_1, y_1), (x_2, y_2)\big) = \|(x_1, y_1) - (x_2, y_2)\|_1 = |x_1 - x_2| + |y_1 - y_2|.$$

It is clear to see that (X, d) is not an uniform convex space. Indeed, for $\varepsilon = 1$, $x = (1, 0)$ and $y = (0, 1)$ we have

$$\|x\|_1 = \|y\|_1 = 1,\ \ \|x - y\|_1 = 2 > 1 = \varepsilon \ \text{ and } \frac{1}{2}\|x + y\|_1 = 1 > 1 - \delta,$$

for each $\delta > 0$. Define the following selfmapping f and g on X:

$$f(0, 0) = f(1, 0) = f(0, 1) = f(1, 1) = (0, 0),$$

$$f(0, 2) = f(2, 0) = (1, 0),$$
$$f(1, 2) = f(2, 1) = f(2, 2) = (0, 1),$$
$$g(0, 0) = g(1, 0) = g(0, 1) = g(1, 1) = g(0, 2) = g(2, 0) = (0, 0),$$
$$g(1, 2) = g(2, 1) = g(2, 2) = (1, 0)$$

and a function $\alpha : X \to \mathbb{R}^+$

$$\alpha(0, 0) = \alpha(1, 0) = \alpha(0, 1) = 0,$$
$$\alpha(1, 1) = \alpha(1, 2) = \alpha(2, 1) = \alpha(0, 2) = \alpha(2, 0) = 1,$$
$$\alpha(2, 2) = 2.$$

Then, $g \circ f = f \circ g$ and for all $x \neq y \in X$, we have

$$\alpha(x) + \alpha(y) + d(fx, gy) - \alpha(fx) - \alpha(gy) - d(fx, gy) \geqslant 1.$$

Hence f and g satisfy all conditions of Theorem such that f and g have the unique common fixed point $(0,0)$.

Remark. The above example illustrates the usability of Theorem 4 and shows that this Theorem is a real extension of Theorem 2.5 in [12], indeed

$$d\big((1,2),(2,1)\big)-d\big(f(1,2),g(2,1)\big)=0$$

or

$$d\big(f(1,2),g(2,1)\big)\leqslant d\big((1,2),f(2,1)\big),$$

which are nonexpansive mappings.

Example 2. Let $X=\overline{B}(0,1)$ be the unit closed ball of a real Banach space $(E,\|\,.\,\|)$. If we take $f=g=0$ and

$$\alpha(x)=\begin{cases}0, & \text{if } x=0,\\ 1, & \text{else.}\end{cases}$$

We observe that f and g satisfy all conditions of Theorem 4 and 0 is the unique common fixed point even if the space X is not compact.

In the following, Φ is the class of all functions $\varphi:[1,+\infty)\longrightarrow[0,+\infty)$ satisfying:

- $\varphi(t)=0$ if and only if $t=1$;
- $\inf\limits_{t>1}\varphi(t)>0$;
- $\varphi(t)\leqslant t$ for all $t\in[1,+\infty)$.

As a consequence of Theorem 4, we get a result for a new class of weakly contractive maps defined as follows:

Definition 8. Let (X,d) be a metric space and $f,g:X\longrightarrow X$ be a two selfmappings of X such that $f\circ g=g\circ f$. f and g are said to be weakly contractive of type E_T, if

$$A(fx,fy)\leqslant A(x,y)-\varphi(A(x,y)+1),$$

where $A(x,y)=\alpha(x)+d(x,y)+\alpha(y)$, for all $x,y\in X$, $\varphi\in\Phi$ and $\alpha:X\longrightarrow[0,+\infty)$ is a bounded and lower semicontinuous function.

Theorem 5. *Let (X,d) be a bounded complete metric space and f,g be two weakly contractive maps of type E_T on X. Then f and g have a unique common fixed point.*

P r o o f. Let $x\neq y\in X$, using Definition 8, we obtain

$$0<\inf_{t>1}\varphi(t)\leqslant\varphi(A(x,y)+1)\leqslant A(x,y)-A(fx,gy),$$

then $\inf_{x\neq y}\{A(x,y)-A(fx,gy)\}>0$. From Theorem, we conclude that f and g have a unique common fixed point. □

Corollary 3. Let (X,d) be a bounded complete metric space. If there exist $i,j\in\mathbb{N}$ such that f^i,f^j be two weakly contractive maps of type E_T on X. Then f has a unique fixed point.

Corollary 4 (Theorem 2.11 [12])**.** Let (X,d) be a bounded complete metric space and f,g be two E_T-weakly contractive maps on X. Then f and g have a unique common fixed point.

Example 3. Let $X=\{1,2\}\cup[3,4]$ with the usual metric $d(x,y)=|x-y|$ for all $x,y\in X$. Define $f,g:X\to X$ by

$$f(x)=\begin{cases}1, & \text{if } x\in\{1,2,3\},\\ 3, & \text{if } x\in(3,4],\end{cases}$$

$$g(x) = \begin{cases} 1, & \text{if } x \in \{1,2,3\}, \\ 2, & \text{if } x \in (3,4], \end{cases}$$

a function $\alpha : X \to \mathbb{R}^+$ defined by

$$\alpha(t) = \begin{cases} 0, & \text{if } t \in \{1,2,3\}, \\ t+2, & \text{if } t \in (3,4], \end{cases}$$

and a function $\varphi : [1,\infty) \to [0,\infty)$ defined by

$$\varphi(t) = \begin{cases} 0, & \text{if } t = 1, \\ 1, & \text{if } t > 1. \end{cases}$$

Therefore, f and g satisfy all assumptions in Theorem 4 and $f1 = g1 = 1$. But the pair (f,g) does not satisfy Theorem 2.11 in [12], indeed

$$d(f4, g3) = 2 > 0 = d(4,3) - \varphi\big(1 + d(4,3)\big).$$

3. Application. Throughout this section we assume that X and Y are Banach spaces. In the language of dynamic programming, $S \subset X$ is the state space and $D \subset Y$ is the decision space. Let $\rho : S \times D \to S$, $g : S \times D \to \mathbb{R}$ and $G_i : S \times D \times \mathbb{R} \to \mathbb{R},\ i = 1,2$. $B(S)$ denotes the set of all bounded real-valued functions on S. For $h, k \in B(S)$, let

$$d(h,k) = \sup\{|h(x) - k(x)| : x \in S\}.$$

It is easy to see that d is a metric on $B(S)$ and $(B(S), d)$ is a complete metric space.

In this section, we study the existence and uniqueness of a common solution of the following class of functional equations arising in dynamic programming:

$$f_i(x) = \sup_{y \in D}\{g(x,y) + G_i(x,y,f_i(\rho(x,y)))\}, \tag{9}$$

where g, G_i are bounded, $i = 1,2$. We define $T_i : B(S) \to B(S)$ by

$$T_i f_i(x) = \sup_{y \in D}\{g(x,y) + G_i(x,y,f_i(\rho(x,y)))\},\ \ i = 1,2. \tag{10}$$

Clearly, T_i are well-defined since g and G_i are bounded.

Suppose that $x \mapsto G_i(.,.,x)$ be nondecreasing and continuous functions such that

$$G_1(x,y,z + G_2(a,b,c)) = G_2(x,y,z + G_1(a,b,c)), \tag{11}$$

for all $x, a \in S,\ y, b \in D$ and $z, c \in \mathbb{R}$.

Now, we prove the existence and uniqueness of a common solution for the system of functional equations (9).

Theorem 6. *Let $T_i : B(S) \to B(S)$ be two operators defined by* (10) *and assume the following conditions are satisfied: there exists $M \in \mathbb{R}^+$ such that*

$$|G_1(x,y,h(v)) - G_2(x,y,k(v))| \leqslant d(h,k),$$

$$g(x,y) + G_1(x,y,h(x)) \leqslant \|h\| - \frac{1}{2}M \tag{12}$$

and

$$g(x,y) + G_2(x,y,k(x)) \leqslant \|k\| - \frac{1}{2}M, \tag{13}$$

for all $(h, k, x, v, y) \in B(S)^2 \times S^2 \times D$, *where* $h(x) \neq k(x)$. *Then the system* (9) *has a unique bounded solution.*

P r o o f. Using condition (11) and the fact that $x \mapsto G_1(.,.,x)$, $x \mapsto G_2(.,.,x)$ are nondecreasing and continuous functions, we get by Lemma 2 that $T_1 \circ T_2 = T_2 \circ T_1$. Now, let λ be an arbitrary positive number, let $x \in S$ and $h, k \in B(S)$, there exist $y, z \in D$ such that

$$T_1 h(x) < g(x, y) + G_1(x, y, h(\rho(x, y))) + \lambda, \tag{14}$$

$$T_2 k(x) < g(x, z) + G_2(x, z, k(\rho(x, z))) + \lambda. \tag{15}$$

On the other hand, by the definition of T_i, we get

$$T_1 h(x) \geqslant g(x, z) + G_1(x, z, h(\rho(x, z))), \tag{16}$$

$$T_2 k(x) \geqslant g(x, y) + G_2(x, y, k(\rho(x, y))). \tag{17}$$

It follows from (14) and (17) that

$$\begin{aligned} T_1 h(x) - T_2 k(x) &< G_1(x, y, h(\rho(x, y))) - G_2(x, y, k(\rho(x, y))) + \lambda, \\ &\leqslant |G_1(x, y, h(\rho(x, y))) - G_2(x, y, k(\rho(x, y)))| + \lambda. \end{aligned}$$

Hence

$$T_1 h(x) - T_2 k(x) \leqslant d(h, k) + \lambda. \tag{18}$$

Similarly from (15) and (16)

$$T_2 k(x) - T_1 h(x) \leqslant d(h, k) + \lambda. \tag{19}$$

In view of (18) and (19), we obtain

$$|T_1 h(x) - T_2 k(x)| \leqslant d(h, k) + \lambda$$

or equivalently

$$d(T_1 h, T_2 k) \leqslant d(h, k) + \lambda.$$

Since λ is taken arbitrary, then we obtain

$$d(T_1 h, T_2 k) \leqslant d(h, k), \tag{20}$$

for all $h \neq k \in B(S)$.

Now, using (12) and (13), we get

$$\alpha(T_1 h) \leqslant \alpha(h) - \frac{1}{2} M \tag{21}$$

and

$$\alpha(T_2 k) \leqslant \alpha(k) - \frac{1}{2} M, \tag{22}$$

where $\alpha(h) = \|h\|$ which is a lower semicontinuous function.

Therefore, we conclude from (20)–(22) that

$$\inf_{h\neq k\in B(S)}\left\{\alpha(h)+d(h,k)+\alpha(k)-\alpha(T_1h)-d(T_1h,T_2k)-\alpha(T_2k)\right\}>0.$$

Finally, we conclude by Theorem 4 that the functional equations (9) has a unique common bounded solution. □

4. Conclusion. Some common fixed point theorems are proved under new weak conditions (without using neither the compactness nor the uniform convexity). Moreover, an application is done to show the utility of the main results.

In addition, the work presented in this document provides tools allowing functional equations in dynamic programming. On the other hand, it opens the way to other future researches:

- extend the proved theorems for the case of four mappings instead of two;
- generalize the results in the setting of generalized orthogonal sets (see [22, 23]).

I would like to express my sincere gratitude to my affiliation "Department of Mathematics, Faculty of Sciences Dhar El Mahraz".

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Received: October 21, 2023.
Accepted: June 25, 2024.

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Об общей неподвижной точке: новые результаты для коммутирующих отображений и их приложение к динамическому программированию

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Для цитирования: *Touail Y.* Common fixed point results: new developments on commuting mappings and application in dynamic programming // Вестник Санкт-Петербургского университета. Прикладная математика. Информатика. Процессы управления. 2024. Т. 20. Вып. 3. С. 366–375. https://doi.org/10.21638/spbu10.2024.305

На основе класса квазинепрерывных функций доказана теорема об общей неподвижной точке для пары коммутирующих отображений. В качестве следствия получена другая общая неподвижная точка для так называемых слабо сжимающих отображений типа E_T. Доказанные результаты установлены в ограниченных метрических пространствах без требования компактности или равномерной выпуклости. Приведены несколько примеров, демонстрирующих преимущество представленных результатов перед опубликованными ранее. Кроме того, рассмотрен пример приложения результатов к системе функциональных уравнений, возникающей в динамическом программировании.

Ключевые слова: общая неподвижная точка, слабо сжимающие отображения типа E_T, коммутирующие отображения, компактность, равномерная выпуклость.

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